J. Differential Equations 246 (2009) 641–669

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

# Global supersonic conic shock wave for the steady supersonic flow past a cone: Polytropic gas ☆

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ARTICLE INFO

*Article history:*
Received 4 March 2008
Revised 16 July 2008
Available online 19 August 2008

*MSC:*
35L70
35L65
35L67
76N15

*Keywords:*
Polytropic gas
Potential flow equation
Supersonic flow
Conic shock
Global existence
Tangent vector field

ABSTRACT

In this paper, we establish the global existence and stability of a steady conic shock wave for the symmetrically perturbed supersonic flow past an infinitely long conic body as long as the vertex angle is less than a critical value. The flow is assumed to be polytropic, isentropic and described by a steady potential equation. Based on the delicate asymptotic expansion of the background solution, one can verify that the boundary conditions on the shock and the conic surface satisfy the “dissipative” property. From this property, by use of the reflected characteristics method and the special form of the shock equation, we show that the conic shock attached at the vertex of the cone exists globally in the whole space when the speed of the supersonic coming flow is appropriately large. On the other hand, we remove the smallness restriction on the sharp vertex angle in order to establish the global existence of a shock or a global weak solution, moreover, our proof approach is different from that in [Shuxing Chen, Zhouping Xin, Huicheng Yin, Global shock wave for the supersonic flow past a perturbed cone, Comm. Math. Phys. 228 (2002) 47–84] and [Zhouping Xin, Huicheng Yin, Global multidimensional shock wave for the steady supersonic flow past a three-dimensional curved cone, Anal. Appl. 4 (2) (2006) 101–132].


☆ This research was supported by the National Natural Science Foundation of China (No. 10571082) and the National Basic Research Program of China (No. 2006CB805902).
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0022-0396/$ – see front matter 
doi:10.1016/j.jde.2008.07.031

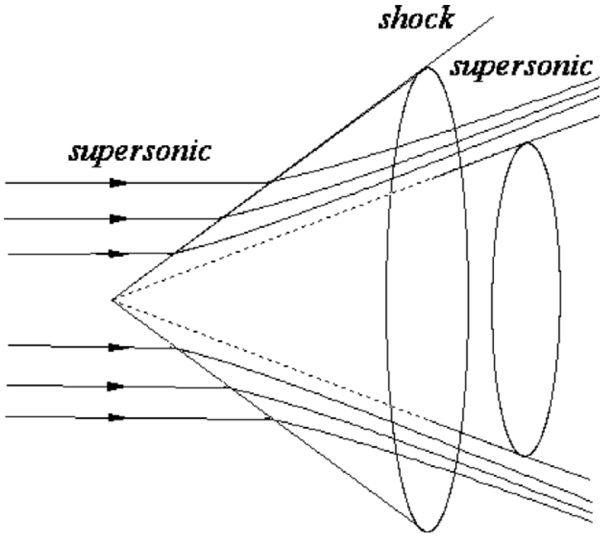


**Fig. 1.**

## 1. Introduction

In this paper we are concerned with the global existence of a shock wave solution for the perturbed steady supersonic gas past an infinitely long conic body when the vertex angle is less than some critical value (see Fig. 1). Such a problem is very important in gas dynamics and has been extensively studied both computationally and experimentally (see [1] and so on). Recently, there have been several interesting results regarding the global existence of the solutions for the uniform supersonic coming flow past a sharp pointed body (see [3,5,9,14,16]). In [3], under the assumptions on the uniform supersonic coming flow with the large Mach number and the sharp angle of the curved conic body, the authors show that a curved conic shock exists globally when the supersonic polytropic flow past a symmetrically curved cone. The so-called polytropic flow means that the pressure $P$ and the density $\rho$ of the gas are described by the state equation $P = A\rho^\gamma$ with $1 < \gamma < 3$. On the other hand, Z. Xin and H. Yin in [14] have established the global existence of a multidimensional conic shock for the uniform supersonic incoming flow with the large Mach number past a generally curved sharp cone under the suitable boundary condition on the conic surface (physically, this kind of boundary condition means that the body is perforated or porous; with respect to more explanations on the perforated boundaries, one can see [6,7]). In addition, by using Glimm's scheme, W.C. Lien and T.P. Liu in [9] obtained the global existence of a weak solution and long distance asymptotic behavior in the symmetric case under the suitable conditions on the Mach number, the sharp vertex angle and the shock strength. Our main interest here is to establish the global existence of a conic shock for the supersonic polytropic gas past an infinitely long conic body as long as the vertex angle is less than some critical value. In particular, we remove the smallness assumption on the sharp cone as mentioned in [3,9,14,16], which is essential in their proofs.

The steady and isentropic compressible Euler systems are described as

$$\begin{cases} \displaystyle\sum_{j=1}^{3} \partial_j(\rho u_j) = 0, \\ \displaystyle\sum_{j=1}^{3} \partial_j(\rho u_i u_j) + \partial_i P = 0, \quad i = 1, 2, 3, \end{cases} \tag{1.1}$$

where $\rho > 0$ denotes the density, $u = (u_1, u_2, u_3)$ the velocity, $P = A\rho^\gamma$ $(1 < \gamma < 3)$ the pressure, $c^2(\rho) = P'(\rho)$ the sound speed, $A > 0$ is a fixed constant.

In this paper, we will restrict ourselves to the irrotational and symmetric case of (1.1). Namely, we will search such a solution of (1.1) with the following form

$$\rho = \rho(x_3, r), \qquad u_1 = U(x_3, r)\frac{x_1}{r}, \qquad u_2 = U(x_3, r)\frac{x_2}{r}, \qquad u_3 = u_3(x_3, r) \tag{1.2}$$

with

$$r = \sqrt{x_1^2 + x_2^2}$$

and

$$\partial_3 U = \partial_r u_3, \qquad \nabla_{x_3, r}\left(\frac{1}{2}\left(U^2 + u_3^2\right) + h(\rho)\right) = 0, \tag{1.3}$$

where $x_3$ is the axis direction of the circular cone $\sqrt{x_1^2 + x_2^2} = b_0 x_3$ ($b_0 > 0$ is a constant), $h(\rho) = \frac{c^2(\rho)}{\gamma - 1}$ the specific enthalpy, $U(x_3, r)$ and $u_3(x_3, r)$ are the components of the fluid velocity in the $r$-direction and $x_3$-direction respectively. The irrotational equation (1.1) with the condition (1.3) is also recommended in [10,11,18].

For this case, the system (1.1) can be reduced to

$$\begin{cases} \partial_r(\rho U) + \partial_3(\rho u_3) + \dfrac{\rho U}{r} = 0, \\ \partial_3 U - \partial_r u_3 = 0, \end{cases} \tag{1.4}$$

where $\rho$ is a function of $U$ and $u_3$ in terms of Bernoulli's law, more concretely, it follows from (1.3) that

$$\rho = h^{-1}\left(C_0 - \frac{1}{2}\left(U^2 + u_3^2\right)\right) \tag{1.5}$$

with $h^{-1}$ the inverse function of $h(\rho)$ and $C_0$ the Bernoulli's constant determined by the steady incoming flow.

In the coordinates $(x_3, r)$, the equation of the cone surface is rewritten as $r = b_0 x_3$. We assume that the equation of the possible shock front attached at the circular cone is denoted by $r = \chi(x_3)$ with $\chi(0) = 0$ (this case can happen when $b_0$ is less than a critical value $b^*$ as indicated in [4]).

Suppose that the flow fields before and behind the shock front $r = \chi(x_3)$ are denoted by $(U^-(x_3, r), u_3^-(x_3, r))$ and $(U^+(x_3, r), u_3^+(x_3, r))$ respectively. Then the system (1.4) can be split into two subsystems, that is, $(U^\pm(x_3, r), u_3^\pm(x_3, r))$ satisfy the following systems in the corresponding domains

$$\begin{cases} \partial_r(\rho^- U^-) + \partial_3\left(\rho^- u_3^-\right) + \dfrac{\rho^- U^-}{r} = 0, \\ \partial_3 U^- = \partial_r u_3^-, \end{cases} \quad r > \chi(x_3),\ x_3 > 0 \tag{1.6}$$

and

$$\begin{cases} \partial_r(\rho^+ U^+) + \partial_3\left(\rho^+ u_3^+\right) + \dfrac{\rho^+ U^+}{r} = 0, \\ \partial_3 U^+ = \partial_r u_3^+, \end{cases} \quad r < \chi(x_3),\ x_3 > 0. \tag{1.7}$$

It is easy to verify that the systems (1.6)–(1.7) are hyperbolic with respect to $x_3$-direction for the supersonic flow ${u_3}^{\pm} > c(\rho^{\pm})$.

On the conic shock front $\Sigma$: $r = \chi(x_3)$, the Rankine–Hugoniot conditions imply

$$\begin{cases} [\rho U] - [\rho u_3]\chi'(x_3) = 0, \\ [u_3] + [U]\chi'(x_3) = 0 \end{cases} \quad \text{on } r = \chi(x_3). \tag{1.8}$$

Moreover, the Lax's geometrical entropy condition should be satisfied (see [13] and so on):

$$\begin{cases} \lambda_1\big(U^+, u_3^+\big)\big(x_3, \chi(x_3) + 0\big) < \chi'(x_3) < \lambda_2\big(U^+, u_3^+\big)\big(x_3, \chi(x_3) + 0\big), \\ \lambda_2\big(U^-, u_3^-\big)\big(x_3, \chi(x_3) - 0\big) < \chi'(x_3) \end{cases} \tag{1.9}$$

with $\lambda_{1,2}(U, u_3) = \frac{u_3 U \mp c(\rho)\sqrt{U^2 + u_3^2 - c^2(\rho)}}{{u_3}^2 - c^2(\rho)}$.

Meanwhile, the fixed boundary condition on the cone surface is

$$U^+ = b_0 u_3^+ \quad \text{on } r = b_0 x_3. \tag{1.10}$$

Additionally, in this paper we assume that the supersonic incoming flow is of a small perturbation of a uniform supersonic flow. More precisely, we pose the following initial conditions on $x_3 = 0$

$$U^-(0, r) = \varepsilon U_0^-(r), \qquad u_3^-(0, r) = q_0 + \varepsilon u_{3,0}^-(r); \tag{1.11}$$

here $(0, 0, q_0)$ and $\rho_0 > 0$ correspond to the velocity and density of a uniform supersonic coming flow with $q_0 > c(\rho_0)$, $\varepsilon > 0$ a small constant, $U_0^-(r), u_{3,0}^-(r) \in C_0^\infty(0, l)$ with some fixed positive number $l > 0$. At this time, the Bernoulli's constant in (1.5) is $C_0 = \frac{1}{2}q_0^2 + h(\rho_0)$, the initial perturbed density $\rho(0, r) = h^{-1}(C_0 - \frac{1}{2}((q_0 + \varepsilon u_{3,0}^-(r))^2 + \varepsilon^2(U_0^-(r))^2))$.

Our main result in this paper is

**Theorem 1.1.** *Suppose that the equation of the circular cone is given by $r = b_0 x_3$ with $0 < b_0 < b_* = \sqrt{\frac{1}{2}\sqrt{\frac{\gamma+7}{\gamma-1}} - 1}$. Then for suitably large $q_0$ and sufficiently small $\varepsilon$ depending on $q_0, b_0, \rho_0$ and $\gamma$, the problem (1.6)–(1.11) admits a global smooth shock solution $(U^{\pm}(x_3, r), u_3^{\pm}(x_3, r); \chi(x_3))$ with $\chi(0) = 0$. Moreover, $(U^+(x_3, r), u_3^+(x_3, r); \chi(x_3))$ tend to the corresponding ones for the uniform supersonic coming flow $(0, 0, q_0; \rho_0)$ past the circular cone $r = b_0 x_3$ with the rate $(1 + x_3)^{-\delta_0}$, where $\delta_0 > 0$ is a suitably small constant independent of $\varepsilon$.*

**Remark 1.1.** By Remark 2.1 in Section 2, we know that there exists a unique conic shock $r = s_0 x_3$ when the uniform supersonic coming flow $(0, 0, q_0; \rho_0)$ hits the cone $r = b_0 x_3$ with $b_0 < b_*$ and $q_0$ is appropriately large. In fact, $b_* = \sqrt{\frac{1}{2}\sqrt{\frac{\gamma+7}{\gamma-1}} - 1}$ is just only the limit of the critical values for the potential equation and hypersonic coming flow when the attached supersonic conic shock phenomenon with $u_3^+(x_3, r) > c(\rho^+(x_3, r))$ happens.

**Remark 1.2.** If the cone is symmetrically curved, by our method in this paper we can show that Theorem 1.1 still holds by use of the local existence result in [2] and [8].

**Remark 1.3.** If the adiabatic exponent $\gamma = 1$ in the state equation, namely, the gas is isothermal, then our analysis on the background solution in Section 2 are not suitable. However, by use of other approaches, we can still obtain the global existence of a shock for the supersonic flow past a cone with an arbitrary vertex angle. This result has been given in [5].

We now comment on the proof of Theorem 1.1. In order to show Theorem 1.1, we need to establish some uniform a priori estimates for the solution and the shock surface of the problem (1.6)–(1.11). Based on such estimates we can use the continuity method for hyperbolic system to obtain the global existence of a shock solution. In [3,14,16], the key ingredients are to look for the suitable multipliers so that the weighted energy estimates on the solution and shock can be derived. Finding such suitable multipliers is rather involved and complicated. Furthermore, the smallness of $b_0$ play a crucial role in finding such multipliers. Thus, for the appropriately large $b_0$, it seems rather difficult for us to use the method in [3,14,16]. In this paper, we intend to use the reflected characteristics method to treat the problem (1.6)–(1.11) as in the wedge case of [12] and [17]. However, since our background solution here is self-similar and not a constant (the wedge case is a constant state, one can see [4] and [17]), then in order to achieve the uniform estimates, we have to overcome the corresponding difficulties induced by the variable coefficient systems and search a new kind of "dissipative" boundary conditions on the shock surface and the conic surface, furthermore, the special form of the shock wave equation will be sufficiently applied. In this procedure, more delicate asymptotic expansions on the background solutions (than those in [3] and [14]) and involved computation on the coefficients of (1.7)–(1.10) are required since these coefficients are closely related to the supersonic coming flow and the cone vertex angle.

Our paper is organized as follows: In Section 2, we derive some elementary estimates on the background solution for the polytropic gas when the speed of the supersonic coming flow is large. In Section 3, first we give a reformulation of the problem (1.6)–(1.11) by introducing the Riemann invariants. Next, based on the results in Section 2 we derive some useful estimates on the coefficients of the reformulated nonlinear system and its boundary conditions. From this, a kind of "dissipative" property on the solution is established in Lemma 3.7. In Section 4, it follows from the preparations in Section 3 that we give the required uniform a priori estimates on the solution and its derivatives. In Section 5, the proof of Theorem 1.1 is completed. Finally, some complicated computations and facts will be given in Appendix A and Appendix B.

In what follows, we will use the following conventions:

$O(q_0^{-\nu})$ $(\nu > 0)$ denotes a bounded quantity, which admits the bound $|O(q_0^{-\nu})| \leqslant M_1 q_0^{-\nu}$, where the generic constant $M_1 > 0$ depend only on $b_0$ and $\gamma$.

$O(\varepsilon)$ means there exist a generic constant $M_2 > 0$, such that $|O(\varepsilon)| \leqslant M_2\varepsilon$, where $M_2 > 0$ depends on $q_0$, $b_0$ and $\gamma$.

## 2. Self-similar solutions and their properties

In the book [4] of R. Courant and K.O. Friedrichs, the following conic shock phenomena for the supersonic flow past a sharp cone are illustrated: Suppose that there is a uniform supersonic flow $(0, 0, q_0)$ with constant density $\rho_0 > 0$ which comes from minus infinity. The flow hits the circular cone in the axis direction. The conic surface is described by $r = b_0 x_3$, then there exists a critical value $b^*$ which is determined by the parameters of the incoming flow such that there will appear a conic shock $r = s_0 x_3$ $(s_0 > b_0)$ attached at the tip for $b_0 < b^*$. Moreover the solution of (1.1) is self-similar, that is, under the cylindrical coordinates $(x_3, r)$, the solution of (1.1) between the shock front and the surface of cone has such a form: $\rho = \rho(s)$, $u_1 = U(s)\frac{x_1}{r}$, $u_2 = U(s)\frac{x_2}{r}$ and $u_3 = u_3(s)$ with $s = \frac{r}{x_3}$. In this case, the system (1.1) can be reduced to a nonlinear ordinary differential system as follows:

$$\begin{cases} \rho'(s) = -\dfrac{\rho U(su_3 - U)}{s((1+s^2)c^2(\rho) - (su_3 - U)^2)}, \\ U'(s) = -\dfrac{c^2(\rho)U}{s((1+s^2)c^2(\rho) - (su_3 - U)^2)}, \quad \text{for } b_0 \leqslant s \leqslant s_0. \\ u_3'(s) = \dfrac{c^2(\rho)U}{(1+s^2)c^2(\rho) - (su_3 - U)^2} \end{cases} \tag{2.1}$$

According to Lemma 2.1 below, we know that the denominator $(1+s^2)c^2(\rho) - (su_3 - U)^2 > 0$ holds for $b_0 \leqslant s \leqslant s_0$. This means that the system (2.1) makes sense.

On the shock front $r = s_0 x_3$, it follows from the Rankine–Hugoniot conditions and Lax's geometric entropy conditions on the 2-shock that

$$\begin{cases} [\rho U] - s_0[\rho u_3] = 0, \\ [u_3] + s_0[U] = 0 \end{cases} \tag{2.2}$$

and

$$\begin{cases} \lambda_1(s_0) < s_0 < \lambda_2(s_0), \\ \dfrac{c(\rho_0)}{\sqrt{q_0^2 - c^2(\rho_0)}} < s_0, \end{cases} \tag{2.3}$$

where $\lambda_{1,2}(s) = \frac{U(s)u_3(s) \mp c(\rho)\sqrt{U^2(s)+u_3^2(s)-c^2(\rho(s))}}{u_3^2(s)-c^2(\rho(s))}$.

Additionally, the flow satisfies the fixed boundary condition on $s = b_0$

$$U(s) = b_0 u_3(s). \tag{2.4}$$

As indicated in [4, pp. 411–414], the boundary value problem (2.1)–(2.4) can be solved by the shooting method or the apple curve picture. More concretely, for any given $b_0 > 0$, which is less than the critical value $b^*$, one can determine the solution of (2.1)–(2.4) by finding the intersection point of the apple curve with the ray $U = b_0 u_3$. Such a solution is called the background solution in this paper.

For the large $q_0$, now we search some critical value $b_*$ and give some precise estimates on the background solution so that we can use them to treat our nonlinear problem in subsequent sections.

**Lemma 2.1.** *If $u_3(b_0) > c(\rho(b_0))$ holds, then the free boundary problem* (2.1)–(2.4) *has a smooth supersonic solution for $b_0 \leqslant s \leqslant s_0$. Moreover, one has*

(i) $U'(s) < 0$, $u_3'(s) > 0$, $\rho'(s) < 0$ *and* $(U^2 + u_3^2 - c^2(\rho))'(s) > 0$.
(ii) $U(s) > 0$, $u_3(s) > c(\rho(s))$ *and* $c^2(\rho(s))(1 + s^2) - (su_3(s) - U(s))^2 > 0$.

**Remark 2.1.** If $q_0$ is sufficiently large, then for any fixed $b_0 < b_* = \sqrt{\frac{1}{2}\sqrt{\frac{\gamma+7}{\gamma-1}} - 1}$, we can verify that $u_3(b_0) > c(\rho(b_0))$ and $u_3(s) > c(\rho(s))$ hold (one can see Remark 2.2 below). Thus, this means that the problem (2.1)–(2.4) has a smooth supersonic shock solution for $1 < \gamma < 3$ and any fixed $b_0$ with $0 < b_0 < b_*$ when $q_0$ is large.

**Proof.** Set $U_+ = \lim_{s \to s_0 - 0} U(s)$, $u_{3+} = \lim_{s \to s_0 - 0} u_3(s)$ and $\rho_+ = \lim_{s \to s_0 - 0} \rho(s)$, then it follows from (2.2) and Bernoulli's law (1.5) that

$$\begin{cases} U_+ = \dfrac{s_0 q_0(\rho_+ - \rho_0)}{(1 + s_0^2)\rho_+}, \\ u_{3+} = q_0 - \dfrac{s_0^2 q_0(\rho_+ - \rho_0)}{(1 + s_0^2)\rho_+}, \\ h(\rho_+) - h(\rho_0) - \dfrac{s_0^2 q_0^2(\rho_+^2 - \rho_0^2)}{2(1 + s_0^2)\rho_+^2} = 0. \end{cases} \tag{2.5}$$

Obviously, $U_+ > 0$ holds due to the entropy condition (2.3).

Since $\frac{s_0u_{3+}-U_+}{\sqrt{1+s_0^2}}$ represents the normal velocity on the shock front, then as in Lemma 2.1 of [3], the entropy condition (2.3) implies

$$s_0u_{3+}-U_+>0, \qquad c(\rho_+)>\frac{s_0u_{3+}-U_+}{\sqrt{1+s_0^2}}. \tag{2.6}$$

By the continuity of $\rho(s), U(s)$ and $u_3(s)$, (2.6) holds in $s_0-\delta\leqslant s\leqslant s_0$ with small $\delta>0$, and then (2.1) makes sense in this interval. Due to (2.1), we know that $\rho'(s)<0, U'(s)<0$ and $u_3'(s)>0$ are valid in $s_0-\delta\leqslant s\leqslant s_0$. In addition, we arrive at

$$\left(c\big(\rho(s)\big)-\frac{su_3(s)-U(s)}{\sqrt{1+s^2}}\right)'=c'\big(\rho(s)\big)\rho'(s)-\frac{su_3'(s)-U'(s)}{\sqrt{1+s^2}}-\frac{u_3(s)+sU(s)}{(1+s^2)^{\frac{3}{2}}}<0.$$

This means that $c(\rho(s))-\frac{su_3(s)-U(s)}{\sqrt{1+s^2}}$ is a decreasing function of $s$. Thus, we can conclude in $s_0-\delta\leqslant s\leqslant s_0$: $U(s)\geqslant U_+$, $\rho(s)\geqslant\rho_+$, and

$$\begin{aligned}c^2\big(\rho(s)\big)\big(1+s^2\big)-\big(su_3(s)-U(s)\big)^2&=\big(1+s^2\big)\left(c\big(\rho(s)\big)-\frac{su_3(s)-U(s)}{\sqrt{1+s^2}}\right)\left(c\big(\rho(s)\big)+\frac{su_3(s)-U(s)}{\sqrt{1+s^2}}\right)\\&\geqslant c(\rho_+)\big(1+b_0^2\big)\left(c(\rho_+)-\frac{s_0u_{3+}-U_+}{\sqrt{1+s_0^2}}\right)>0.\end{aligned} \tag{2.7}$$

One can derive from (2.7) that the denominator in (2.1) is bounded away from zero as long as the solution of (2.1) exists. Therefore, (2.7) holds in the whole interval $[b_0,s_0]$, and the solution of (2.1) exists there, which satisfies

$$U'(s)<0, \qquad u_3'(s)>0, \qquad \rho'(s)<0.$$

Moreover by a direct computation, we have

$$\big(u_3(s)-c\big(\rho(s)\big)\big)'=\frac{sc^2(\rho(s))U(s)+c'(\rho(s))\rho(s)U(s)(su_3(s)-U(s))}{s((1+s^2)c^2(\rho)-(su_3-U)^2)}>0,$$

and this yields

$$u_3(s)-c\big(\rho(s)\big)>u_3(b_0)-c\big(\rho(b_0)\big)>0.$$

Namely, (2.1)–(2.4) has a supersonic solution with $u_3(s)>c(\rho(s))$ if $u_3(b_0)>c(\rho(b_0))$. Then we complete the proof of Lemma 2.1. □

Next, for large $q_0$, we list some useful estimates on the background solution which have been given in Lemma 2.2 and Lemma 2.3 of [3].

**Lemma 2.2.** *If $q_0$ is large, for $1<\gamma<3$, $0<b_0<b_*$ and $b_0\leqslant s\leqslant s_0$, then*

(i) $s_0=b_0+O(q_0^{-\frac{2}{\gamma-1}})$.
(ii) $0\leqslant su_3(s)-U(s)\leqslant O(q_0^{\frac{\gamma-3}{\gamma-1}})$.
(iii) $U(s)=\frac{b_0q_0}{1+b_0^2}+O(q_0^{\frac{\gamma-3}{\gamma-1}})$.

(iv) $u_3(s)=\frac{q_0}{1+b_0^2}+O(q_0^{\frac{\gamma-3}{\gamma-1}})$.

(v) $\rho(s)=(\frac{(\gamma-1)b_0^2}{2A\gamma(1+b_0^2)})^{\frac{1}{\gamma-1}}q_0^{\frac{2}{\gamma-1}}(1+O(q_0^{-2})+O(q_0^{-\frac{2}{\gamma-1}}))$.

(vi) $q^2(s)-c^2(\rho(s))=q_0^2(\frac{2-(\gamma-1)b_0^2}{2(1+b_0^2)})(1+O(q_0^{-2})+O(q_0^{-\frac{2}{\gamma-1}}))$; *here and below* $q^2(s)=U^2(s)+u_3^2(s)$.

(vii) $u_3(s)-c(\rho(s))=\frac{q_0}{1+b_0^2}(1-b_0\sqrt{\frac{\gamma-1}{2}}\sqrt{1+b_0^2})+O(q_0{}^{-1})+O(q_0^{\frac{\gamma-3}{\gamma-1}})>0$.

(viii) $U'(s)=-\frac{q_0}{(1+b_0^2)^2}+O(q_0{}^{-1})+O(q_0^{\frac{\gamma-3}{\gamma-1}})$.

(ix) $u_3'(s)=\frac{b_0q_0}{(1+b_0^2)^2}+O(q_0{}^{-1})+O(q_0^{\frac{\gamma-3}{\gamma-1}})$.

(x) $|\rho'(s)|\leqslant C$.

**Remark 2.2.** It follows from (vii) of Lemma 2.2 that the assumption $u_3(b_0)>c(\rho(b_0))$ in Lemma 2.1 holds when $q_0$ is large and $b_0<b_*=\sqrt{\frac{1}{2}\sqrt{\frac{\gamma+7}{\gamma-1}}-1}$. In fact, $b_*$ is determined by the algebraic equation $1-b_*\sqrt{\frac{\gamma-1}{2}}\sqrt{1+b_*^2}=0$.

One can see the proofs of Lemma 2.2 and Lemma 2.3 of [3], here we omit them.

Based on Lemma 2.2, we can derive the estimates on the second order derivatives of $U(s)$ and $u_3(s)$ for $b_0\leqslant s\leqslant s_0$.

**Lemma 2.3.** *Under the assumptions of Lemma* 2.2*, one has*

(i) $U''(s)=\frac{2+3b_0^2}{b_0(1+b_0^2)^3}q_0(1+O(q_0{}^{-2})+O(q_0^{-\frac{2}{\gamma-1}}))$.

(ii) $u_3''(s)=-\frac{1+2b_0^2}{(1+b_0^2)^3}q_0(1+O(q_0{}^{-2})+O(q_0^{-\frac{2}{\gamma-1}}))$.

**Proof.** (i) It follows from (2.1) and a direct computation that

$$\begin{aligned}
U''(s)=&-\frac{2c(\rho)c'(\rho)\rho'U+c^2(\rho)U'}{s((1+s^2)c^2(\rho)-(su_3-U)^2)}+\frac{c^2(\rho)U}{s^2((1+s^2)c^2(\rho)-(su_3-U)^2)}\\
&+\frac{c^2(\rho)U(2c(\rho)c'(\rho)\rho'(1+s^2)+2sc^2(\rho)-2(su_3-U)(u_3+su_3'-U'))}{s((1+s^2)c^2(\rho)-(su_3-U)^2)^2}\\
=&\left(-\frac{U'}{b_0(1+b_0^2)}+\frac{U}{b_0^2(1+b_0^2)}+\frac{2U}{(1+b_0^2)^2}\right)\left(1+O\left(q_0{}^{-1}\right)+O\left(q_0^{\frac{\gamma-3}{\gamma-1}}\right)\right)\\
=&\frac{2+3b_0^2}{b_0(1+b_0^2)^3}q_0\left(1+O\left(q_0{}^{-2}\right)+O\left(q_0^{-\frac{2}{\gamma-1}}\right)\right).
\end{aligned}$$

(ii) Similarly, one has

$$\begin{aligned}
u_3''&=\left(\frac{U'}{1+b_0^2}+\frac{2b_0U}{(1+b_0^2)^2}\right)\left(1+O\left(q_0{}^{-1}\right)+O\left(q_0^{\frac{\gamma-3}{\gamma-1}}\right)\right)\\
&=-\frac{1+2b_0^2}{(1+b_0^2)^3}q_0\left(1+O\left(q_0{}^{-2}\right)+O\left(q_0^{-\frac{2}{\gamma-1}}\right)\right). \qquad\square
\end{aligned}$$

In order to prove Theorem 1.1, we need to give more precise estimates on the background solution than those in Lemma 2.2. For the simplicity to write, we will introduce the notations $y_0$ and $y_1$ as follows:

$$y_0=\left(\frac{2A\gamma(1+s_0^2)}{\rho_0^{1-\gamma}(\gamma-1)s_0^2}\right)^{\frac{1}{\gamma-1}}q_0^{-\frac{2}{\gamma-1}},\qquad y_1=\frac{2h(\rho_0)(1+b_0^2)}{b_0^2}q_0^{-2}.$$

**Lemma 2.4.** *If $q_0$ is large, for $1<\gamma<3$ and $b_0<b_*$, one has*

(i) $s_0=b_0+\frac{b_0(1+b_0^2)}{2}y_0+O(q_0^{-\frac{2\gamma}{\gamma-1}})+O(q_0^{-\frac{4}{\gamma-1}})$.
(ii) $U_+=\frac{b_0q_0}{1+b_0^2}(1-\frac{1+b_0^2}{2}y_0+O(q_0^{-\frac{2\gamma}{\gamma-1}})+O(q_0^{-\frac{4}{\gamma-1}}))$.
(iii) $u_{3+}=\frac{q_0}{1+b_0^2}(1+O(q_0^{-\frac{2\gamma}{\gamma-1}})+O(q_0^{-\frac{4}{\gamma-1}}))$.
(iv) $c^2(\rho_+)=\frac{(\gamma-1)b_0^2}{2(1+b_0^2)}q_0^2(1+y_0+y_1+O(q_0^{-\frac{2\gamma}{\gamma-1}})+O(q_0^{-\frac{4}{\gamma-1}}))$.
(v) $q_+^2-c^2(\rho_+)=\frac{2-(\gamma-1)b_0^2}{2(1+b_0^2)}q_0^2(1-\frac{(\gamma+1)b_0^2}{2-(\gamma-1)b_0^2}y_0-\frac{(\gamma-1)b_0^2}{2-(\gamma-1)b_0^2}y_1+O(q_0^{-\frac{2\gamma}{\gamma-1}})+O(q_0^{-\frac{4}{\gamma-1}}))$; *here* $q_+^2=U_+^2+(u_{3+})^2$.

**Proof.** (i) From the third equation in (2.5), we have on $s=s_0$

$$\frac{A\gamma}{\gamma-1}\left(\rho_+^{\gamma-1}-\rho_0^{\gamma-1}\right)=\frac{s_0^2q_0^2}{2(1+s_0^2)}\left(1-\left(\frac{\rho_0}{\rho_+}\right)^2\right).$$

Set $\Lambda=\frac{\rho_+}{\rho_0}$, then one has

$$\Lambda^{\gamma-1}=1+\frac{s_0^2\rho_0^{1-\gamma}(\gamma-1)q_0^2}{2A\gamma(1+s_0^2)}\left(1-\frac{1}{\Lambda^2}\right).$$

This implies

$$\Lambda=\left(\frac{s_0^2\rho_0^{1-\gamma}(\gamma-1)}{2A\gamma(1+s_0^2)}\right)^{\frac{1}{\gamma-1}}q_0^{\frac{2}{\gamma-1}}\left(1+O\left(q_0^{-2}\right)\right).\tag{2.8}$$

Let $m=\frac{s_0^2\rho_0^{1-\gamma}(\gamma-1)}{2A\gamma(1+s_0^2)}$, then (2.8) is equivalent to

$$\frac{1}{\Lambda}=m^{-\frac{1}{\gamma-1}}q_0^{-\frac{2}{\gamma-1}}+O\left(q_0^{-\frac{2\gamma}{\gamma-1}}\right).\tag{2.9}$$

Thus, it follows from (2.5) that

$$\begin{cases} U_+ = \dfrac{s_0 q_0}{1+s_0^2}\left(1-\dfrac{1}{\Lambda}\right), \\ u_{3+} = \dfrac{q_0}{1+s_0^2}\left(1+\dfrac{s_0^2}{\Lambda}\right), \\ q_+^2 = \dfrac{q_0^2}{1+s_0^2}\left(1+\dfrac{s_0^2}{\Lambda^2}\right), \\ c^2(\rho_+) = (\gamma-1)\left(\dfrac{s_0^2 q_0^2}{2(1+b_0^2)}\left(1-\dfrac{1}{\Lambda^2}\right)+h(\rho_0)\right), \\ s_0 u_{3+} - U_+ = \dfrac{s_0 q_0}{\Lambda}. \end{cases} \tag{2.10}$$

Substituting (2.10) into (2.1) yields

$$\begin{cases} U'(s_0) = -\dfrac{q_0}{(1+s_0^2)^2}\left(1-\dfrac{1}{\Lambda}+O\left(\dfrac{1}{\Lambda^2}\right)\right), \\ u_3'(s_0) = -s_0 U'(s_0) = \dfrac{s_0 q_0}{(1+s_0^2)^2}\left(1-\dfrac{1}{\Lambda}+O\left(\dfrac{1}{\Lambda^2}\right)\right). \end{cases} \tag{2.11}$$

By using the condition $U(b_0) = b_0 u_3(b_0)$, Taylor's formula, Lemma 2.2 and Lemma 2.3, we can obtain

$$s_0 - b_0 = \frac{b_0 u_{3+} - U_+}{b_0 u_3'(s_0) - U'(s_0)} + O\left(q_0^{-\frac{4}{\gamma-1}}\right). \tag{2.12}$$

Combining (2.10), (2.11) with (2.12) yields

$$s_0 - b_0 = \frac{b_0(1+b_0^2)}{2\Lambda} + O\left(q_0^{-\frac{4}{\gamma-1}}\right) = \frac{b_0(1+b_0^2)}{2} y_0 + O\left(q_0^{-\frac{2\gamma}{\gamma-1}}\right) + O\left(q_0^{-\frac{4}{\gamma-1}}\right).$$

(ii), (iii), (iv) and (v) comes from (i) and (2.10). $\square$

**Remark 2.3.** Since the denominator of the system (2.1) is positive in $[b_0, s_0]$, by use of the system (2.1) we can extend the background solution $(\rho(s), u_3(s), U(s))$ of (2.1)–(2.4) to the interval $[b_0, s_0+\tau_0]$ for small positive constant $\tau_0$ satisfying $0 < \tau_0 \leqslant q_0^{-\frac{4}{\gamma-1}}(s_0 - b_0)$. In the following sections we will denote the extension of the background solution in the domain $\{(x_3, r)\colon\ x_3 > 0,\ b_0 x_3 \leqslant r \leqslant (s_0+\tau_0)x_3\}$ by $(\hat{\rho}(s), \hat{u}_3(s), \hat{U}(s))$ with $s = \frac{r}{x_3}$.

## 3. The reformulation of problem (1.6)–(1.11) and some useful estimates

In this section, first we introduce the Riemann invariants to reformulate the problem (1.7)–(1.10) so that we can derive a $2\times 2$ diagonal system and the resulted nonlinear boundary conditions on the conic shock front and the fixed boundary. Next, based on the analysis on the background solution in Section 2, we can give the needed estimates about the coefficients which appear in the reformulated problem when $q_0$ is large. In particular, we can show that the resulted boundary conditions are "dissipative" together with the shock equation. This essential ingredient will play a crucial role in obtaining the uniform estimates of solution in the domain $\{(x_3, r)\colon\ b_0 \leqslant r \leqslant \chi(x_3),\ x_3 \geqslant t_0\}$ with some fixed $t_0 > 0$.

We now give a global existence result of solution to (1.6) and (1.11) in the left-hand side of the shock.

**Lemma 3.1.** *The system* (1.6) *with the initial data* (1.11) *has a $C^\infty$ solution $(U^-(x_3,r), u_3^-(x_3,r))$ in the domain $\Omega_- = \{(x_3,r)\colon x_3 \geqslant 0,\ r \geqslant \chi(x_3)\}$. Moreover, $(U^-(x_3,r), u_3^-(x_3,r) - q_0) \in C_0^\infty(\Omega_-)$, and there exists a positive constant $C_k$ independent of $\varepsilon$ such that*

$$\left\|U^-(x_3,r)\right\|_{C^k(\Omega_-)} + \left\|u_3^-(x_3,r) - q_0\right\|_{C^k(\Omega_-)} \leqslant C_k\varepsilon$$

*for any fixed $k \in \mathbb{N}$.*

**Proof.** We note that the system (1.6) is quasi-linear strictly hyperbolic with respect to the $x_3$-direction for the supersonic flow $u_3^- > c(\rho^-)$, furthermore, the initial condition (1.11) is of a small perturbation with compact support. Thus, in terms of the entropy condition (2.3), the finite propagation property of the hyperbolic systems and the Picard iteration (or one can see [8]), we know that Lemma 3.1 holds. □

Next, we start to reformulate the nonlinear problem (1.7)–(1.10). For notational convenience, we will neglect all the superscripts "+" in (1.7)–(1.10).

The system (1.7) has two distinct real eigenvalues

$$\lambda_1(U,u_3) = \frac{u_3U - c\sqrt{q^2-c^2}}{{u_3}^2 - c^2} \quad \text{and} \quad \lambda_2(U,u_3) = \frac{u_3U + c\sqrt{q^2-c^2}}{{u_3}^2 - c^2}$$

with $q^2 = U^2 + u_3^2$ and $c = c(\rho(q^2))$.

Corresponding to $\lambda_1(U,u_3)$ and $\lambda_2(U,u_3)$, one can introduce two Riemann invariants as follows

$$\begin{cases} \omega_1(x_3,r) = \arctan\dfrac{U}{u_3} + F(q), \\ \omega_2(x_3,r) = \arctan\dfrac{U}{u_3} - F(q), \end{cases} \tag{3.1}$$

where $F'(q) = \frac{\sqrt{q^2-c^2}}{qc}$.

It follows from the system (1.7) and a direct computation that $\omega = (\omega_1,\omega_2)$ satisfies

$$\begin{cases} \partial_{x_3}\omega_1 + \lambda_1(\omega)\partial_r\omega_1 = \dfrac{f_1(\omega)}{r}, \\ \partial_{x_3}\omega_2 + \lambda_2(\omega)\partial_r\omega_2 = \dfrac{f_2(\omega)}{r}, \end{cases} \tag{3.2}$$

where $\lambda_i(\omega) = \lambda_i(U(\omega), u_3(\omega))$, $f_1(\omega) = \frac{cU}{cU + u_3\sqrt{q^2-c^2}}$, $f_2(\omega) = \frac{cU}{cU - u_3\sqrt{q^2-c^2}}$, $U(\omega)$ and $u_3(\omega)$ denote the inverse functions of transformation (3.1).

By (1.8), we have on $r = \chi(x_3)$

$$\begin{cases} \rho(\omega)U^2(\omega) + \big(\rho(\omega)u_3(\omega) - \rho_0q_0\big)\big(u_3(\omega) - q_0\big) = 0, \\ \chi'(x_3) = -\dfrac{u_3(\omega) - q_0}{U(\omega)}. \end{cases} \tag{3.3}$$

While on the circular cone surface $r = b_0x_3$, one has

$$b_0u_3(\omega) - U(\omega) = 0. \tag{3.4}$$

Since the solution of (1.7)–(1.10) will be expected to be a small perturbations of the background solution $(\hat U(s), \hat u_3(s))$ in Section 2, then it is convenient to use $W_i(x_3,r) = \omega_i(x_3,r) - \hat\omega_i(\frac{r}{x_3})$ as the unknown functions instead of $\omega_i(x_3,r)$; here $\hat{\omega}_1(s)=\arctan\frac{\hat{U}(s)}{\hat{u}_3(s)}+F(\sqrt{\hat{U}^2(s)+\hat{u}_3^2(s)})$ and $\hat{\omega}_2(s)=\arctan\frac{\hat{U}(s)}{\hat{u}_3(s)}-F(\sqrt{\hat{U}^2(s)+\hat{u}_3^2(s)})$.

A direct computation yields

$$\begin{cases}\partial_{x_3}W_1+\lambda_1(\omega)\partial_r W_1=g_1(W),\\ \partial_{x_3}W_2+\lambda_2(\omega)\partial_r W_2=g_2(W)\end{cases}\tag{3.5}$$

with

$$\begin{aligned}g_i(W)={}&\frac{1}{r}\left(f_i\big(\omega(x_3,r)\big)-f_i\left(\hat{\omega}\left(\frac{r}{x_3}\right)\right)\right)\\&+\left(\lambda_i\left(\hat{\omega}\left(\frac{r}{x_3}\right)\right)-\lambda_i\big(\omega(x_3,r)\big)\right)\partial_r\hat{\omega}_i\left(\frac{r}{x_3}\right),\quad i=1,2.\end{aligned}\tag{3.6}$$

Next, we reformulate the free boundary condition (3.3) and the fixed boundary condition (3.4) under the transformation (3.1).

On the free boundary $r=\chi(x_3)$, we will introduce the following notation

$$\xi(x_3)=\frac{\chi(x_3)-s_0x_3}{x_3},$$

which describes the perturbation of the slope of the shock front.

By using Taylor's formula and implicit function theorem, then it follows the first equation in (3.3) and (2.2) that on $r=\chi(x_3)$

$$W_1(x_3,r)=A(s_0)W_2(x_3,r)+A_1(s_0)\xi(x_3)+\kappa\big(\xi(x_3),W_2(x_3,r)\big),\tag{3.7}$$

where

$$\begin{aligned}A(s)&=-\frac{\partial_{\omega_2}U(s)}{\partial_{\omega_1}U(s)}\cdot\frac{m_1(s)-\lambda_2(s)m_2(s)}{m_1(s)-\lambda_1(s)m_2(s)},\\
A_1(s)&=-\frac{\rho'(s)U^2(s)+2\rho(s)U(s)U'(s)+(\rho(s)u_3(s)-\rho_0q_0)u_3'(s)}{m_1(s)\partial_{\omega_1}U(s)+m_2(s)\partial_{\omega_1}u_3(s)}\\
&\quad-\frac{(u_3(s)-q_0)(\rho'(s)u_3(s)+\rho(s)u_3'(s))}{m_1(s)\partial_{\omega_1}U(s)+m_2(s)\partial_{\omega_1}u_3(s)},\\
m_1(s)&=-\frac{\rho(s)U^3(s)}{c^2(\rho(s))}+2\rho(s)U(s)-\frac{\rho(s)U(s)u_3(s)}{c^2(\rho(s))}\big(u_3(s)-q_0\big),\\
m_2(s)&=-\frac{\rho(s)u_3(s)U^2(s)}{c^2(\rho(s))}+\rho(s)\big(u_3(s)-q_0\big)-\frac{\rho(s)u_3^2(s)}{c^2(\rho(s))}\big(u_3(s)-q_0\big)+\rho(s)u_3(s)-\rho_0q_0\end{aligned}$$

and $\kappa(0,0)=0$, $\kappa\in C^\infty$ on its arguments.

In what follows, the generic function $\kappa(\xi(x_3),W(x_3,r))$ will be used to denote any quantity dominated by $C(|\xi(x_3)|^2+|W(x_3,r)|^2)$; here the generic constant $C>0$ does not depend on $\varepsilon$.

In addition, it follows from the second equation in (3.3) and (2.2) that on $r=\chi(x_3)$

$$x_3\xi'(x_3)=A_2(s_0)\xi(x_3)+l_1(s_0)W_1(x_3,r)+l_2(s_0)W_2(x_3,r)+\kappa\big(\xi(x_3),W(x_3,r)\big),\tag{3.8}$$

where

$$A_2(s)=\frac{(u_3(s)-q_0)U'(s)-U(s)u_3'(s)}{U^2(s)}-1,$$
$$l_1(s)=\frac{(u_3(s)-q_0)\partial_{\omega_1}U(s)-U(s)\partial_{\omega_1}u_3(s)}{U^2(s)},$$
$$l_2(s)=\frac{(u_3(s)-q_0)\partial_{\omega_2}U(s)-U(s)\partial_{\omega_2}u_3(s)}{U^2(s)}.$$

It can be derived from (3.8) that

$$\big(x_3^{-A_2(s_0)}\xi(x_3)\big)'=x_3^{-A_2(s_0)-1}\big(l_1(s_0)W_1(x_3,r)+l_2(s_0)W_2(x_3,r)+\kappa\big(\xi(x_3),W(x_3,r)\big)\big). \tag{3.9}$$

Similarly, by using Taylor's formula and the implicit function theorem, we can rewrite the boundary condition (3.4) as follows

$$W_2(x_3,r)=B(b_0)W_1(x_3,r)+f_3\big(W_1(x_3,r)\big)\quad \text{on } r=b_0x_3, \tag{3.10}$$

where

$$B(s)=-\frac{\partial_{\omega_1}U(s)}{\partial_{\omega_2}U(s)}\cdot\frac{1+b_0\lambda_1(s)}{1+b_0\lambda_2(s)}$$

and $f_3(0)=f_3'(0)=0$, $f_3$ is smooth on its argument.

Since it follows from Lemma 3.1 that the solution $(U^-(x_3,r),u_3^-(x_3,r)-q_0)\in C_0^\infty(\Omega_-)$, then near the vertex of the cone $r=b_0x_3$, the solution $(U^+(x_3,r),u_3^+(x_3,r);\chi(x_3))$ is actually the background solution $(U(s),u_3(s);s_0)$ with $s=\frac{r}{x_3}$. In order to prove Theorem 1.1, by the local existence result in [8] or [13] for $x_3\geqslant t_0$ with some fixed constant $t_0>0$, we only need to solve the problem (3.2) with the boundary conditions (3.7), (3.8), (3.10) and the small initial data of $W_i(x_3,r)|_{x_3=t_0}$ $(i=1,2)$ and $\xi(x_3)|_{x_3=t_0}$ in the domain $\{(x_3,r)\colon\ x_3\geqslant t_0,\ b_0x_3\leqslant r\leqslant\chi(x_3)\}$. Here the smallness means that

$$\sum_{|\alpha|\leqslant 1}\sup_{b_0t_0\leqslant r\leqslant\chi(t_0)}\big|\nabla_{x_3,r}^\alpha W_i(t_0,r)\big|\leqslant C\varepsilon,\qquad \big|\xi(t_0)\big|+\big|\xi'(t_0)\big|+\big|\xi''(t_0)\big|\leqslant C\varepsilon, \tag{3.11}$$

where $\varepsilon$ is sufficiently small and is given in Theorem 1.1. In addition, we notice that (3.11) can be derived from Lemma 3.1 and the local existence and stability results in [8].

For the later use, we require the precise estimates on the coefficients in (3.7), (3.8) and (3.10) so that we can derive the "dissipative" property of the system (3.2) together with the shock equation (3.9).

**Lemma 3.2.** *For $m_1(s_0)$ and $m_2(s_0)$ in (3.7), we have*

$$\frac{m_2(s_0)}{m_1(s_0)}=\frac{1-b_0^2}{2b_0}\left(1+\frac{1+b_0^2}{2(1-b_0^2)}\left(\frac{3-\gamma}{\gamma-1}-b_0^2\right)y_0\right)+O\big(q_0^{-\frac{4}{\gamma-1}}\big)+O\big(q_0^{-\frac{2\gamma}{\gamma-1}}\big).$$

**Proof.** In light of the R–H conditions (2.2), one has

$$\rho_+u_{3+}-\rho_0q_0=\frac{\rho_+U_+}{s_0}. \tag{3.12}$$

Substituting (3.12) into the expression of $m_2(s_0)$ and using Lemma 2.4 yields

$$
\begin{aligned}
m_2(s_0) &= \rho_+\left(-\frac{u_{3+}q_+^2}{c^2(\rho_+)} + u_{3+} - q_0 + \frac{q_0 u_{3+}^2}{c^2(\rho_+)} + \frac{U_+}{s_0}\right) \\
&= \frac{\rho_+ q_0}{1+b_0^2}\left(-\frac{2}{(\gamma-1)b_0^2}\cdot\frac{1-b_0^2 y_0}{1+y_0+y_1} - b_0^2 + \frac{2}{(\gamma-1)b_0^2}\cdot\frac{1}{1+y_0+y_1} + \frac{2-(1+b_0^2)y_0}{2+(1+b_0^2)y_0}\right. \\
&\quad \left. + O\big(q_0^{-\frac{2\gamma}{\gamma-1}}\big) + O\big(q_0^{-\frac{4}{\gamma-1}}\big)\right).
\end{aligned}
$$

In addition, a direct computation also yields

$$
\begin{aligned}
m_1(s_0) &= \rho_+\left(-\frac{U_+ q_+^2}{c^2(\rho_+)} + 2U_+ + \frac{q_0 U_+ u_{3+}}{c^2(\rho_+)}\right) \\
&= \frac{\rho_+ q_0}{1+b_0^2}\left(-\frac{1}{(\gamma-1)b_0}\cdot\frac{(2-(1+b_0^2)y_0)(1-b_0^2 y_0)}{1+y_0+y_1} + b_0\big(2-\big(1+b_0^2\big)y_0\big)\right. \\
&\quad \left. + \frac{1}{(\gamma-1)b_0}\cdot\frac{2-(1+b_0^2)y_0}{1+y_0+y_1} + O\big(q_0^{-\frac{2\gamma}{\gamma-1}}\big) + O\big(q_0^{-\frac{4}{\gamma-1}}\big)\right).
\end{aligned}
$$

Thus we derive that

$$
\frac{m_2(s_0)}{m_1(s_0)} = \frac{1-b_0^2}{2b_0}\left(1 + \frac{1+b_0^2}{2(1-b_0^2)}\left(\frac{3-\gamma}{\gamma-1} - b_0^2\right)y_0\right) + O\big(q_0^{-\frac{4}{\gamma-1}}\big) + O\big(q_0^{-\frac{2\gamma}{\gamma-1}}\big). \qquad \square
$$

**Lemma 3.3.** *We have*

$$
A_2(s_0) = -1.
$$

**Proof.** It follows from $u_3{}'(s_0) = -s_0 U'(s_0)$ that

$$
A_2(s_0) = \frac{U'(s_0)(s_0 U_+ + u_{3+} - q_0)}{U_+^2} - 1.
$$

Thus, by use of the second equation in (2.2) we obtain $A_2(s_0) = -1$. $\square$

Based on Lemma 3.2, we can obtain the following result:

**Lemma 3.4.** *If $1 < \gamma < 3$ and $0 < b_0 < b_*$, then for large $q_0$, one has*

$$
A(s_0) = -\frac{A_{01}(s_0)}{A_{02}(s_0)} + O\big(q_0^{-\frac{2\gamma}{\gamma-1}}\big) + O\big(q_0^{-\frac{4}{\gamma-1}}\big) \tag{3.13}
$$

*with*

$$
\begin{aligned}
A_{01}(s_0) &= 1 - \frac{2-\gamma}{\gamma-1}y_0 - \left(1 + \frac{1}{2}\left(\frac{3-\gamma}{\gamma-1} - 3b_0^2\right)y_0\right)\frac{c_+}{b_0\sqrt{q_+^2 - c_+^2}}, \\
A_{02}(s_0) &= 1 - \frac{2-\gamma}{\gamma-1}y_0 + \left(1 + \frac{1}{2}\left(\frac{3-\gamma}{\gamma-1} - 3b_0^2\right)y_0\right)\frac{c_+}{b_0\sqrt{q_+^2 - c_+^2}}.
\end{aligned}
$$

*Roughly speaking,*

$$A(s_0) = -\frac{\sqrt{1-\frac{\gamma-1}{2}b_0^2}-\sqrt{\frac{\gamma-1}{2}}}{\sqrt{1-\frac{\gamma-1}{2}b_0^2}+\sqrt{\frac{\gamma-1}{2}}} + O\big(q_0^{-2}\big) + O\big(q_0^{-\frac{2}{\gamma-1}}\big). \tag{3.14}$$

**Remark 3.1.** In order to verify some "dissipative" property in Lemma 3.7 below for the boundary conditions (3.7), (3.8) and (3.10), we require the precise form of $A(s_0)$ in (3.13). One can see the explanations in subsequent Remark 3.4.

The proof of Lemma 3.4 will be given in Appendix A.

**Lemma 3.5.** *For large $q_0$, $1<\gamma<3$ and $b_0<b_*$, one has*

$$\big|A_1(s_0)\big|\big(\big|l_1(s_0)\big|+\big|l_2(s_0)\big|\big) = \frac{2}{Q(s_0)} + O\big(q_0^{-\frac{2\gamma}{\gamma-1}}\big) + O\big(q_0^{-\frac{4}{\gamma-1}}\big) \tag{3.15}$$

*with*

$$Q(s_0) = 1 + \left(1+\frac{3+b_0^2}{2}y_0\right)b_0\sqrt{\frac{q_+^2-c^2(\rho_+)}{c^2(\rho_+)}} + \left(\frac{2}{\gamma-1}-b_0^2\right)y_0.$$

*This roughly derives*

$$\big|A_1(s_0)\big|\big(\big|l_1(s_0)\big|+\big|l_2(s_0)\big|\big) = \frac{2\sqrt{\frac{\gamma-1}{2}}}{\sqrt{1-\frac{\gamma-1}{2}b_0^2}+\sqrt{\frac{\gamma-1}{2}}} + O\big(q_0^{-2}\big) + O\big(q_0^{-\frac{2}{\gamma-1}}\big). \tag{3.16}$$

**Remark 3.2.** As in Remark 3.1, we also need the precise form of $|A_1(s_0)|(|l_1(s_0)|+|l_2(s_0)|)$ in (3.15). The related explanations will be given in Remark 3.4.

One can see the proof of Lemma 3.5 in Appendix A.
Next, we estimate the coefficient $B(b_0)$ in (3.10).

**Lemma 3.6.** *We have*

$$B(b_0) = -1.$$

Its proof is given in Appendix A.
It follows from the preparations in Lemmas 3.2–3.6 that we can establish the following crucial results.

**Lemma 3.7.** *If $1<\gamma<3$ and $0<b_0<b_*$, then for large $q_0$, we have*

(i) $$0 < \big|A(s_0)B(b_0)\big| < 1. \tag{3.17}$$

(ii) *There exists a suitably small positive constant $\delta_0$ depending on $q_0$ such that*

$$0 < \big|A(s_0)B(b_0)\big| + \frac{|A_1(s_0)||B(b_0)|(|l_1(s_0)|+|l_2(s_0)|)}{1-\delta_0} < 1. \tag{3.18}$$

**Remark 3.3.** We point out that the relations (3.17) and (3.18) actually correspond to the "dissipative" boundary conditions on the shock and the fixed boundary together with the shock equation (3.9). (3.17) and (3.18) will play a key role in obtaining the uniform decay estimates of $W(x_3,r)$ and its derivative in the subsequent Section 4. Similar idea can be used to show the global existence and stability in time when a transonic shock lies in a diverging part of a nozzle. Moreover, if the condition (3.18) is violated, the shock solution will blow up in finite time (when a transonic shock lies in a converging part of a nozzle, this case will really happen). One can see more details in [15].

**Proof.** (i) By use of (3.14) and Lemma 3.6, it is easy to verify that

$$0<\left|A(s_0)B(b_0)\right|<1$$

holds for large $q_0$.

(ii) It follows from Lemma 2.4(iv) and (v) that

$$\frac{b_0\sqrt{q_+^2-c^2(\rho_+)}}{c(\rho_+)}=(1+\beta)\sqrt{\frac{2-(\gamma-1)b_0^2}{\gamma-1}}+O\big(q_0^{-\frac{2\gamma}{\gamma-1}}\big)+O\big(q_0^{-\frac{4}{\gamma-1}}\big)$$

with

$$\beta=\sqrt{\frac{2-(\gamma-1)b_0^2-(\gamma+1)b_0^2y_0-(\gamma-1)b_0^2y_1}{(2-(\gamma-1)b_0^2)(1+y_0+y_1)}}-1.$$

Then in view of (3.13), (3.15) and Lemma 3.6, a direct computation yields

$$\begin{aligned}
&\left|A(s_0)B(b_0)\right|+\left|A_1(s_0)\right|\left|B(b_0)\right|\big(\left|l_1(s_0)\right|+\left|l_2(s_0)\right|\big)\\
&=1-\frac{2\sqrt{\frac{\gamma-1}{2}}+y_0(\frac{3-\gamma}{\gamma-1}-3b_0^2)\sqrt{\frac{\gamma-1}{2}}}{\sqrt{1-\frac{\gamma-1}{2}b_0^2}+\sqrt{\frac{\gamma-1}{2}}+y_0\big(\frac{1}{2}(\frac{3-\gamma}{\gamma-1}-3b_0^2)\sqrt{\frac{\gamma-1}{2}}-\frac{2-\gamma}{\gamma-1}\sqrt{1-\frac{\gamma-1}{2}b_0^2}\big)+\beta\sqrt{1-\frac{\gamma-1}{2}b_0^2}(1-\frac{2-\gamma}{\gamma-1}y_0)}\\
&\quad+\frac{2\sqrt{\frac{\gamma-1}{2}}}{\sqrt{1-\frac{\gamma-1}{2}b_0^2}+\sqrt{\frac{\gamma-1}{2}}+y_0(\frac{3+b_0^2}{2}\sqrt{1-\frac{\gamma-1}{2}b_0^2}+(\frac{2}{\gamma-1}-b_0^2)\sqrt{\frac{\gamma-1}{2}})+\beta\sqrt{1-\frac{\gamma-1}{2}b_0^2}(1+\frac{3+b_0^2}{2}y_0)}\\
&\quad+O\big(q_0^{-\frac{2\gamma}{\gamma-1}}\big)+O\big(q_0^{-\frac{4}{\gamma-1}}\big)\\
&=1+\frac{y_0\sqrt{\frac{\gamma-1}{2}}(\sqrt{1-\frac{\gamma-1}{2}b_0^2}+\sqrt{\frac{\gamma-1}{2}})(-\frac{4}{\gamma-1}+2b_0^2)}{(\sqrt{1-\frac{\gamma-1}{2}b_0^2}+\sqrt{\frac{\gamma-1}{2}})^2+O(q_0^{-2})+O(q_0^{-\frac{2}{\gamma-1}})}+O\big(q_0^{-\frac{2\gamma}{\gamma-1}}\big)+O\big(q_0^{-\frac{4}{\gamma-1}}\big)\\
&=1-y_0\frac{2-(\gamma-1)b_0^2}{\sqrt{\frac{\gamma-1}{2}}(\sqrt{1-\frac{\gamma-1}{2}b_0^2}+\sqrt{\frac{\gamma-1}{2}})}+O\big(q_0^{-\frac{2\gamma}{\gamma-1}}\big)+O\big(q_0^{-\frac{4}{\gamma-1}}\big).
\end{aligned}\tag{3.19}$$

Hence, for $1<\gamma<3$, $b_0<b_*$ and large $q_0$, there exists a small positive constant $\delta_0$ depending on $q_0, b_0$ and $\gamma$ such that

$$\left|A(s_0)B(b_0)\right|+\frac{|A_1(s_0)||B(b_0)|(|l_1(s_0)|+|l_2(s_0)|)}{1-\delta_0}<1. \qquad \square$$

**Remark 3.4.** To verify the term $|A(s_0)B(b_0)|+|A_1(s_0)||B(b_0)|(|l_1(s_0)|+|l_2(s_0)|)<1$ in (3.19), we have to use (3.13) and (3.15). If we apply for the rough forms (3.14) and (3.16), then one can only obtain

$$\begin{aligned}\big|A(s_0)B(b_0)\big|+\big|A_1(s_0)\big|\big|B(b_0)\big|\big(\big|l_1(s_0)\big|+\big|l_2(s_0)\big|\big) &= 1-y_0\frac{2-(\gamma-1)b_0^2}{\sqrt{\frac{\gamma-1}{2}}\big(\sqrt{1-\frac{\gamma-1}{2}b_0^2}+\sqrt{\frac{\gamma-1}{2}}\big)}+O\big(q_0^{-2}\big)\\ &\quad+O\big(q_0^{-\frac{2\gamma}{\gamma-1}}\big)+O\big(q_0^{-\frac{4}{\gamma-1}}\big). \end{aligned}\tag{3.20}$$

In this case, for $1<\gamma<\frac{3}{2}$, we cannot assert that $|A(s_0)B(b_0)|+|A_1(s_0)||B(b_0)|(|l_1(s_0)|+|l_2(s_0)|)<1$ holds since $y_0\ll q_0^{-2}$ for large $q_0$ and it is difficult to determine the sign of term $O(q_0^{-2})$ in the right-hand side of (3.20).

## 4. Uniform estimates on $W(x_3,r)$, $\xi(x_3)$ and their derivatives

In this section, we will derive the uniform decay estimates on $W(x_3,r),\xi(x_3)$ and their derivatives. Using the estimates, we can easily show Theorem 1.1 in Section 5.

**Theorem 4.1** *(Uniform decay estimates). Let $D_T=\{(x_3,r);\ t_0\leqslant x_3\leqslant T,\ b_0x_3\leqslant r\leqslant\chi(x_3)\}$ for any large $T>t_0$. If $W(x_3,r)\in C^2(D_T)$ satisfies* (3.5), (3.7)–(3.8) *and* (3.10)–(3.11), *then for large $q_0$, there exist two positive constants $C_0$ and $\tilde C_0$ independent of $\varepsilon$ and $T$, such that $|\partial_{x_3,r}^{\beta}W_i(x_3,r)|\leqslant\frac{C_0\varepsilon}{(1+x_3)^{\delta_0+|\beta|}}$ in $D_T$ for $|\beta|\leqslant1$, $i=1,2$, and $|\frac{d^j}{dx_3^j}\xi(x_3)|\leqslant\frac{\tilde C_0\varepsilon}{(1+x_3)^{\delta_0+j}}$ in $[t_0,T]$ for $j=0,1,2$; here $\delta_0$ is given in* (3.18).

**Proof.** We shall use the reflected characteristics method together with the special form of the shock equation (3.9) to obtain the needed estimates. By the local existence result in [8] and the continuity induction, we only need to prove:

*For some positive suitable constants $C_m$ $(0\leqslant m\leqslant4)$, if $|W_i|\leqslant\frac{C_0\varepsilon}{(1+x_3)^{\delta_0}}$ and $|\nabla_{x_3,r}W_i|\leqslant\frac{C_1\varepsilon}{(1+x_3)^{\delta_0+1}}$ in $D_T$; $|\xi(x_3)|\leqslant\frac{C_2\varepsilon}{(1+x_3)^{\delta_0}}$, $|\xi'(x_3)|\leqslant\frac{C_3\varepsilon}{(1+x_3)^{1+\delta_0}}$ and $|\xi''(x_3)|\leqslant\frac{C_4\varepsilon}{(1+x_3)^{2+\delta_0}}$ in $[t_0,T]$, then there exist positive constants $C_m'$ with $C_m'<C_m$ such that*

$$\begin{gathered}|W_i|\leqslant\frac{C_0'\varepsilon}{(1+x_3)^{\delta_0}},\qquad|\nabla_{x_3,r}W_i|\leqslant\frac{C_1'\varepsilon}{(1+x_3)^{\delta_0+1}}\quad\text{in } D_T;\\ \big|\xi(x_3)\big|\leqslant\frac{C_2'\varepsilon}{(1+x_3)^{\delta_0}},\qquad\big|\xi'(x_3)\big|\leqslant\frac{C_3'\varepsilon}{(1+x_3)^{1+\delta_0}}\quad\textit{and}\quad\big|\xi''(x_3)\big|\leqslant\frac{C_4'\varepsilon}{(1+x_3)^{2+\delta_0}}\quad\text{in }[t_0,T].\end{gathered}\tag{4.1}$$

For $(x_3,r)\in D_T$, we shall denote by $\gamma_j(s,x_3,r)$ $(j=1,2)$ the backward $j$th characteristic curve passing the point $(x_3,r)$, that is

$$\begin{cases}\dfrac{d\gamma_j(t,x_3,r)}{dt}=\lambda_j\big(\omega\big(t,\gamma_j(t,x_3,r)\big)\big), & t\leqslant x_3,\\ \gamma_j(t,x_3,r)|_{t=x_3}=r.\end{cases}\tag{4.2}$$

By the assumptions in (4.1) and Lemma B.1 in Appendix B, we arrive at

$$\left|\frac{d\gamma_j(t,x_3,r)}{dt}-\lambda_j\big(\omega(s)\big)\right|\leqslant\frac{(C+O(q_0^{-1})+O(\varepsilon))C_0\varepsilon}{(1+t)^{\delta_0}}\quad\text{in } D_T;\tag{4.3}$$

here and below the generic constant $C>0$ is independent of $q_0$ and $\varepsilon$.

If $(t,\gamma_1(t,x_3,r))\cap\{(t,z)\colon\ z=\chi(t)\}=(\Gamma_1(x_3,r),\xi_1(x_3,r))$ and $(t,\gamma_2(t,x_3,r))\cap\{(t,z)\colon\ z=b_0t\}=(\Gamma_2(x_3,r),\xi_2(x_3,r))$, then from (B.1) in Appendix B and the system (3.5), one has for large $q_0$

$$\begin{aligned}\left|W_i(x_3,r)\right| &\leqslant \left|W_i\big(\Gamma_i(x_3,r),\xi_i(x_3,r)\big)\right| \\ &\quad + \big(C+O\big(q_0^{-1}\big)+O(\varepsilon)\big)\int\limits_{\Gamma_i(x_3,r)}^{x_3}\sum_{j=1}^{2}\frac{1}{t}\left|W_j\big(t,\gamma_i(t,x_3,r)\big)\right|dt. \end{aligned}\tag{4.4}$$

If $(t,\gamma_1(t,\Gamma_2(x_3,r),\xi_2(x_3,r)))\cap\{(t,z)\colon\ z=\chi(t)\}=(\pi_1(x_3,r),\eta_1(x_3,r))$ and $(t,\gamma_2(t,\Gamma_1(x_3,r)),\xi_1(x_3,r))\cap\{(t,z)\colon\ z=b_0t\}=(\pi_2(x_3,r),\eta_2(x_3,r))$, then for large $q_0$ and sufficient small $\varepsilon$, by using the characteristics method and the boundary condition (3.7) and (3.10), we obtain

$$\begin{aligned}\left|W_2(x_3,r)\right| &\leqslant \left|W_2\big(\Gamma_2(x_3,r),\xi_2(x_3,r)\big)\right| + \big(C+O\big(q_0^{-1}\big)+O(\varepsilon)\big)\int\limits_{\Gamma_2(x_3,r)}^{x_3}\frac{1}{t}\sum_{j=1}^{2}\left|W_j\big(t,\gamma_2(t,x_3,r)\big)\right|dt \\
&\leqslant \left|B(b_0)+O(\varepsilon)\right|\Bigg(\left|W_1\big(\pi_1(x_3,r),\eta_1(x_3,r)\big)\right| \\
&\quad + \big(C+O\big(q_0^{-1}\big)+O(\varepsilon)\big)\int\limits_{\pi_1(x_3,r)}^{\Gamma_2(x_3,r)}\frac{1}{t}\sum_{j=1}^{2}\left|W_j\big(t,\gamma_1\big(t,\Gamma_2(x_3,r),\xi_2(x_3,r)\big)\big)\right|dt\Bigg) \\
&\quad + \big(C+O\big(q_0^{-1}\big)+O(\varepsilon)\big)\int\limits_{\Gamma_2(x_3,r)}^{x_3}\frac{1}{t}\sum_{j=1}^{2}\left|W_j\big(t,\gamma_2(t,x_3,r)\big)\right|dt \\
&\leqslant \left|B(b_0)+O(\varepsilon)\right|\big(\left|A(s_0)\right|\left|W_2\big(\pi_1(x_3,r),\eta_1(x_3,r)\big)\right| + \left|A_1(s_0)\right|\left|\xi\big(\pi_1(x_3,r)\big)\right| + O\big(\varepsilon^2\big)\big) \\
&\quad + \big(C+O\big(q_0^{-1}\big)+O(\varepsilon)\big)\left|B(b_0)+O(\varepsilon)\right|\int\limits_{\pi_1(x_3,r)}^{\Gamma_2(x_3,r)}\frac{1}{t}\sum_{j=1}^{2}\left|W_j\big(t,\gamma_1\big(t,\Gamma_2(x_3,r),\xi_2(x_3,r)\big)\big)\right|dt \\
&\quad + \big(C+O\big(q_0^{-1}\big)+O(\varepsilon)\big)\int\limits_{\Gamma_2(x_3,r)}^{x_3}\frac{1}{t}\sum_{j=1}^{2}\left|W_j\big(t,\gamma_2(t,x_3,r)\big)\right|dt. \end{aligned}\tag{4.5}$$

In an analogous way, we can derive that

$$\begin{aligned}\left|W_1(x_3,r)\right| &\leqslant \big(\left|A(s_0)B(b_0)\right|+O(\varepsilon)\big)\left|W_1\big(\pi_2(x_3,r),\eta_2(x_3,r)\big)\right| + \big(\left|A_1(s_0)\right|+O(\varepsilon)\big)\left|\xi_1\big(\Gamma_1(x_3,r)\big)\right| \\
&\quad + C\left|A(s_0)+O(\varepsilon)\right|\int\limits_{\pi_2(x_3,r)}^{\Gamma_1(x_3,r)}\frac{1}{t}\sum_{j=1}^{2}\left|W_j\big(t,\gamma_2\big(t,\Gamma_1(x_3,r),\xi_1(x_3,r)\big)\big)\right|dt \\
&\quad + C\int\limits_{\Gamma_1(x_3,r)}^{x_3}\frac{1}{t}\sum_{j=1}^{2}\left|W_j\big(t,\gamma_1(t,x_3,r)\big)\right|dt. \end{aligned}\tag{4.6}$$

By Lemma B.3 in Appendix B, for small $\varepsilon$ and large $q_0$, we have

$$d_i - C\varepsilon \leqslant \frac{\Gamma_i(x_3,r)}{x_3} \leqslant 1+C\varepsilon, \qquad d_1d_2 - C\varepsilon \leqslant \frac{\pi_i(x_3,r)}{x_3} \leqslant d_i + C\varepsilon, \quad i=1,2; \tag{4.7}$$

here $0<d_1<1$, $0<d_2<1$, $\frac{1}{d_1}<1+Cq_0^{-\frac{2}{\gamma-1}}$ and $\frac{1}{d_1d_2}<1+Cq_0^{-\frac{2}{\gamma-1}}$.

This, together with the assumptions in (4.1), yields

$$\begin{aligned}\left|W_2\big(\pi_1(x_3,r),\eta_1(x_3,r)\big)\right| &\leqslant \frac{C_0\varepsilon}{(1+x_3)^{\delta_0}}\cdot\frac{1+C\varepsilon}{(d_1d_2)^{\delta_0}}\\ &\leqslant \frac{C_0\varepsilon(1+C\delta_0 q_0^{-\frac{2}{\gamma-1}}+C\varepsilon)}{(1+x_3)^{\delta_0}}.\end{aligned} \tag{4.8}$$

In addition, it follows from (3.9) and Lemma 3.3 that

$$\left|\big(x_3\xi(x_3)\big)'\right| \leqslant \frac{(|l_1(s_0)|+|l_2(s_0)|)C_0\varepsilon}{(1+x_3)^{\delta_0}}+\frac{C(C_0+C)^2\varepsilon^2}{(1+x_3)^{2\delta_0}}.$$

Thus, for $x_3>t_0$ and small $\varepsilon$, by use of (3.11) we have

$$\left|\xi(x_3)\right| \leqslant \frac{C\varepsilon}{1+x_3}+\frac{(|l_1(s_0)|+|l_2(s_0)|+O(\varepsilon))C_0\varepsilon}{(1+x_3)^{\delta_0}(1-\delta_0)}. \tag{4.9}$$

Substituting (4.7)–(4.9) into (4.5) yields for large $q_0$ and $x_3>t_0$

$$\begin{aligned}\left|W_2(x_3,r)\right| \leqslant{}& \left|A(s_0)B(b_0)+O(\varepsilon)\right|\left|W_2\big(\pi_1(x_3,r),\eta_1(x_3,r)\big)\right|\\ &+\left|A_1(s_0)B(b_0)+O(\varepsilon)\right|\left|\xi\big(\pi_1(x_3,r)\big)\right|+C\int_{\pi_1(x_3,r)}^{x_3}\frac{C_0\varepsilon}{(1+t)^{1+\delta_0}}\,dt\\ \leqslant{}& \left(\left|A(s_0)B(b_0)\right|+\frac{|A_1(s_0)B(b_0)|(|l_1(s_0)|+|l_2(s_0)|)}{1-\delta_0}+O(\varepsilon)\right)\frac{1+C\varepsilon}{(d_1d_2)^{\delta_0}}\cdot\frac{C_0\varepsilon}{(1+x_3)^{\delta_0}}\\ &+\big(\left|A_1(s_0)B(b_0)\right|+O(\varepsilon)\big)\frac{1+C\varepsilon}{d_1d_2}\cdot\frac{C_0\varepsilon}{1+x_3}+\frac{C\varepsilon}{\delta_0(1+x_3)^{\delta_0}}\left(\frac{1+C\varepsilon}{(d_1d_2)^{\delta_0}}-1\right)\\ \leqslant{}& \left(\left|A(s_0)B(b_0)\right|+\frac{|A_1(s_0)B(b_0)|(|l_1(s_0)|+|l_2(s_0)|)}{1-\delta_0}+O(\varepsilon)\right)\frac{C_0\varepsilon(1+C\delta_0q_0^{-\frac{2}{\gamma-1}}+O(\varepsilon))}{(1+x_3)^{\delta_0}}\\ &+\big(\left|A_1(s_0)B(b_0)\right|+O(\varepsilon)\big)\frac{C\varepsilon(1+Cq_0^{-\frac{2}{\gamma-1}}+O(\varepsilon))}{1+x_3}+\frac{C\varepsilon(Cq_0^{-\frac{2}{\gamma-1}}+O(\varepsilon))}{(1+x_3)^{\delta_0}};\end{aligned} \tag{4.10}$$

here we have used the very useful fact $\frac{1}{\delta_0}(\frac{1}{(d_1d_2)^{\delta_0}}-1)\leqslant Cq_0^{-\frac{2}{\gamma-1}}$ with the constant $C>0$ independent of $q_0$.

Similarly, we have the same estimate on $|W_1(x_3,r)|$.

If either of the following four cases holds

$$\begin{aligned}\big(t,\gamma_1(t,x_3,r)\big)\cap\big\{(t,z):\ z=\chi(t)\big\}&=\emptyset,\\ \big(t,\gamma_2(t,x_3,r)\big)\cap\big\{(t,z):\ z=b_0t\big\}&=\emptyset,\\ \big(t,\gamma_1\big(t,\Gamma_2(x_3,r),\xi_2(x_3,r)\big)\big)\cap\big\{(t,z):\ z=\chi(t)\big\}&=\emptyset,\\ \big(t,\gamma_2\big(t,\Gamma_1(x_3,r)\big),\xi_1(x_3,r)\big)\cap\big\{(t,z):\ z=b_0t\big\}&=\emptyset,\end{aligned}$$

then by (4.3) and the initial data (3.11) we can conclude for the suitable constant $\tilde{C}_0$:

$$x_3 \leqslant \tilde{C}_0 \quad \text{and} \quad \left|W_i(x_3, r)\right| \leqslant \tilde{C}_0 \varepsilon. \tag{4.11}$$

Thus, for large $q_0$, $x_3$ and small $\varepsilon$, it follows from (3.18) and (4.10)–(4.11) that there exist two suitable positive constants $C_0$ and $C_0'$ with $C_0' < C_0$ such that

$$\left|W_1(x_3, r)\right| \leqslant \frac{C_0'\varepsilon}{(1+x_3)^{\delta_0}}, \qquad \left|W_2(x_3, r)\right| \leqslant \frac{C_0'\varepsilon}{(1+x_3)^{\delta_0}} \quad \text{in } D_T. \tag{4.12}$$

In addition, for large $q_0$, (4.9) implies that there exist a suitable constant $C_2 > 0$ and a positive constant $C_2'$ with $C_2' < C_2$ such that

$$\left|\xi(x_3)\right| \leqslant \frac{C_2'\varepsilon}{(1+x_3)^{\delta_0}}. \tag{4.13}$$

Next we estimate $\nabla_{x_3,r} W_i(x_3, r)$.

Denote by $Z = \partial_{x_3} + b_0\partial_r$ the tangent vector of the boundary $r = b_0 x_3$. Set $\tilde{W}_i = Z W_i(x_3, r)$, $i = 1, 2$; then by (3.5) we obtain

$$\begin{cases} \partial_{x_3}\tilde{W}_1 + \lambda_1(\omega)\partial_r\tilde{W}_1 = \tilde{g}_1(W), \\ \partial_{x_3}\tilde{W}_2 + \lambda_2(\omega)\partial_r\tilde{W}_2 = \tilde{g}_2(W), \end{cases} \tag{4.14}$$

where

$$\begin{aligned}
\tilde{g}_1(W) = {} & \frac{1}{r} Z\big(f_1(\omega) - f_1(\hat{\omega})\big) - \frac{b_0}{r^2}\big(f_1(\omega) - f_1(\hat{\omega})\big) - \frac{\hat{\omega}_1'}{x_3} Z\big(\lambda_1(\omega) - \lambda_1(\hat{\omega})\big) \\
& - \big(\lambda_1(\omega) - \lambda_1(\hat{\omega})\big)\left(-\frac{1}{x_3^2}\left(\hat{\omega}_1' + \hat{\omega}_1''\frac{r}{x_3}\right) + \frac{b_0}{x_3^2}\hat{\omega}_1''\right) - \partial_r W_1\big(Z\big(\lambda_1(\omega) - \lambda_1(\hat{\omega})\big) + Z\lambda_1(\hat{\omega})\big), \\
\tilde{g}_2(\omega) = {} & \frac{1}{r} Z\big(f_2(\omega) - f_2(\hat{\omega})\big) - \frac{b_0}{r^2}\big(f_2(\omega) - f_2(\hat{\omega})\big) - \frac{\hat{\omega}_2'}{x_3} Z\big(\lambda_2(\omega) - \lambda_2(\bar{\omega})\big) \\
& - \big(\lambda_2(\omega) - \lambda_2(\hat{\omega})\big)\left(-\frac{1}{x_3^2}\left(\hat{\omega}_2' + \hat{\omega}_2''\frac{r}{x_3}\right) + \frac{b_0}{x_3^2}\hat{\omega}_2''\right) - \partial_r W_2\big(Z\big(\lambda_2(\omega) - \lambda_2(\hat{\omega})\big) + Z\lambda_2(\hat{\omega})\big).
\end{aligned}$$

By the assumptions in (4.1), Lemma B.1 and Lemma B.2 of Appendix B, we have

$$\left|\tilde{g}_i(W)\right| \leqslant \frac{C(C_0 + C_1 + O(q_0^{-\frac{2}{\gamma-1}}) + O(\varepsilon))\varepsilon}{(1+x_3)^{2+\delta_0}}. \tag{4.15}$$

From (3.10), one has

$$\tilde{W}_2 = B(b_0)\tilde{W}_1 + f_3'(W_1)\tilde{W}_1 \quad \text{on } r = b_0 x_3. \tag{4.16}$$

To get the boundary condition of $\tilde{W}$ on the shock front $r = \chi(x_3)$, one should note that the vector field $V = \partial_{x_3} + \chi'(x_3)\partial_r$ which is tangent to $r = \chi(x_3)$ can be expressed as follows

$$V = \frac{1}{\lambda_i(\omega) - b_0}\Big(\big(\lambda_i(\omega) - \chi'(x_3)\big)Z + \big(\chi'(x_3) - b_0\big)\big(\partial_3 + \lambda_i(\omega)\partial_r\big)\Big).$$

Thus, on the shock $r = \chi(x_3)$, from (3.5) and (3.7) we have

$$\begin{aligned}\tilde{W}_1 &= \big(A(s_0)+O(\varepsilon)\big)\bigg(\frac{(\lambda_2(\omega)-\chi'(x_3))(b_0-\lambda_1(\omega))}{(\lambda_2(\omega)-b_0)(\chi'(x_3)-\lambda_1(\omega))}\tilde{W}_2+\frac{(\chi'(x_3)-b_0)(b_0-\lambda_1(\omega))}{(\lambda_2(\omega)-b_0)(\chi'(x_3)-\lambda_1(\omega))}g_2\bigg)\\ &\quad+\big(A_1(s_0)+O(\varepsilon)\big)\frac{\lambda_1(\omega)-b_0}{\lambda_1(\omega)-\chi'(x_3)}\xi'(x_3)+\frac{\chi'(x_3)-b_0}{\chi'(x_3)-\lambda_1(\omega)}g_1\\ &= A(s_0)\frac{(\lambda_2(\omega)-\chi'(x_3))(b_0-\lambda_1(\omega))}{(\lambda_2(\omega)-b_0)(\chi'(x_3)-\lambda_1(\omega))}\tilde{W}_2+\tilde{\kappa}\big(\tilde{W}_2,\xi'(x_3)\big);\end{aligned} \tag{4.17}$$

here

$$\begin{aligned}\big|\tilde{\kappa}\big(\tilde{W}_2,\xi'(x_3)\big)\big| &\leqslant C\varepsilon|\tilde{W}_2|+\big(\big|A_1(s_0)\big|+O(\varepsilon)+O\big(q_0^{-\frac{2}{\gamma-1}}\big)\big)\big|\xi'(x_3)\big|\\ &\quad+\frac{(O(q_0^{-\frac{2}{\gamma-1}})+O(\varepsilon))\varepsilon}{(1+x_3)^{1+\delta_0}}.\end{aligned} \tag{4.18}$$

Additionally, by (3.9), (4.9) and (4.12), for large $q_0$ and appropriate large $x_3$ one has

$$\begin{aligned}\big|\xi'(x_3)\big| &\leqslant \frac{C\varepsilon}{x_3(1+x_3)}+\left(1+\frac{1}{1-\delta_0}\right)\big(\big|l_1(s_0)\big|+\big|l_2(s_0)\big|+O(\varepsilon)\big)\frac{C_0\varepsilon}{x_3(1+x_3)^{\delta_0}}\\ &\leqslant \frac{C\varepsilon}{(1+x_3)^2}+\frac{CC_0\varepsilon}{(1+x_3)^{1+\delta_0}}.\end{aligned} \tag{4.19}$$

Under the assumptions of (4.1), we assume that there exist two suitable positive constants $\tilde{C}_1$ such that $|\tilde{W}_i|\leqslant\frac{\tilde{C}_1\varepsilon}{(1+x_3)^{1+\delta_0}}$ in $D_T$. Then, by using the characteristics method together with the estimates (4.15) and (4.18) and Lemma B.3 of Appendix B, we can obtain for large $q_0$:

$$\begin{aligned}\big|\tilde{W}_1(x_3,r)\big| &\leqq \big|\tilde{W}_1\big(\Gamma_1(x_3,r),\xi_1(x_3,r)\big)\big|+\int_{\Gamma_1(x_3,r)}^{x_3}\big|\tilde{g}_1\big(t,\gamma_1(t,x_3,r)\big)\big|\,dt\\ &\leqq \big|A(s_0)\big|\big|\tilde{W}_2\big(\Gamma_1(x_3,r),\xi_1(x_3,r)\big)\big|+\big|\tilde{\kappa}\big(\tilde{W}_2,\xi'\big(\Gamma_1(x_3,r)\big)\big)\big|+\int_{\Gamma_1(x_3,r)}^{x_3}\big|\tilde{g}_1\big(t,\gamma_1(t,x_3,r)\big)\big|\,dt\\ &\leqslant \big(\big|A(s_0)B(b_0)\big|+O(\varepsilon)\big)\big|\tilde{W}_1\big(\pi_2(x_3,r),\eta_2(x_3,r)\big)\big|+\big(\big|A(s_0)\big|+O(\varepsilon)\big)\int_{\pi_2(x_3,r)}^{\Gamma_1(x_3,r)}|\tilde{g}_2|\,ds\\ &\quad+\frac{(O(q_0^{-\frac{2}{\gamma-1}})+O(\varepsilon))\varepsilon}{(1+\Gamma_1(x_3,r))^{1+\delta_0}}+\big(\big|A_1(s_0)\big|+O(\varepsilon)+O\big(q_0^{-\frac{2}{\gamma-1}}\big)\big)\big|\xi'\big(\Gamma_1(x_3,r)\big)\big|+\int_{\Gamma_1}^{x_3}|\tilde{g}_1|\,ds\\ &\leqslant \big(\big|A(s_0)B(b_0)\big|+O(\varepsilon)\big)\frac{\tilde{C}_1\varepsilon(1+O(\varepsilon)+O(q_0^{-\frac{2}{\gamma-1}}))}{(1+x_3)^{1+\delta_0}}\\ &\quad+\frac{(1+O(\varepsilon)+O(q_0^{-\frac{2}{\gamma-1}}))CC_0\varepsilon}{(1+x_3)^{1+\delta_0}}.\end{aligned} \tag{4.20}$$

Similarly, we have the similar estimate on $\tilde{W}_2(x_3,r)$.

Noting that the generic constant $C$ in (4.20) is independent of $\tilde{C}_1$, then for large $q_0$ and $x_3$, by use of (3.17) of Lemma 3.7 we can choose two constants $\tilde{C}_1$ and $\tilde{C}_1'$ with $\tilde{C}_1' < \tilde{C}_1$ such that

$$|\tilde{W}_1| \leqslant \frac{\tilde{C}_1\varepsilon}{(1+x_3)^{1+\delta_0}}, \qquad |\tilde{W}_2| \leqslant \frac{\tilde{C}_1\varepsilon}{(1+x_3)^{1+\delta_0}};$$

here $\tilde{C}_1$ depend only on $C_0$, $b_0$ and $\gamma$.

In addition, it follows from (3.5) and the definition of $\tilde{W}_i$ that

$$\partial_3 W_i(x_3,r) = \frac{\lambda_i(\omega)\tilde{W}_i - b_0 g_i}{\lambda_i(\omega) - b_0}, \qquad \partial_r W_i(x_3,r) = \frac{g_i - \tilde{W}_i}{\lambda_i(\omega) - b_0}. \tag{4.21}$$

By a direction computation, one can show that there exist two constants $C_1$ and $C_1'$ with $C_1' < C_1$ such that

$$\left|\nabla_{x_3,r} W_i(x_3,r)\right| \leqslant \frac{C_1'\varepsilon}{(1+x_3)^{1+\delta_0}} \quad \text{in } D_T; \tag{4.22}$$

here $C_1 > 0$ depends on $C_0$.

By (4.12)–(4.13), (4.19) and (4.22), it follows from Eq. (3.8) that there exist two constants $C_4 > 0$ and $C_4' > 0$ with $C_4' < C_4$ such that

$$\left|\xi''(x_3)\right| \leqslant \frac{C_4\varepsilon}{(1+x_3)^{2+\delta_0}};$$

here $C_4$ depends on $C_0$ and $C_1$.

Thus we complete the proof of (4.1). Namely, Theorem 4.1 is proved. □

## 5. Proof of Theorem 1.1

Based on the uniform decay estimates of $W_i(x_3,r)$ $(i=1,2)$, $\xi(x_3)$ and their derivatives in Theorem 4.1, we can show the global existence of a conic shock solution in Theorem 1.1.

Indeed, the local existence of the solution to Eq. (3.5) with (3.7)–(3.8) and (3.10)–(3.11) can be achieved by use of the result in [8], namely, for any given $t_0 > 0$, the $C^2$-solution exists uniquely in $[t_0, t_0+\eta]$ for some fixed $\eta > 0$. Furthermore, by the smallness of the perturbed initial data on $x_3 = t_0$, we know that the lifespan of the shock solution is at least as large as $\frac{M}{\varepsilon}$ with $M > 0$. By the uniform estimates in Theorem 4.1, the local existence result and the standard continuity extension method, one can obtain the global existence of a $C^2$ shock solution. Since the initial-boundary values are $C^\infty$, then the regularity of solution can be improved to be $C^\infty$. Thus, Theorem 1.1 is proved.

## Appendix A

In this appendix, we now give proofs of Lemma 3.4, Lemma 3.5 and Lemma 3.6.

**Proof of Lemma 3.4.** Firstly, by the expression of $A(s_0)$, a direct computation yields

$$A(s_0) = -\frac{G_1(s_0)}{G_2(s_0)}$$

with

$$G_1(s_0)=\left(u_{3+}-\frac{U_+c(\rho_+)}{\sqrt{q_+^2-c^2(\rho_+)}}\right)\left\{\left(1-\frac{u_{3+}U_+}{u_{3+}^2-c^2(\rho_+)}\frac{m_2(s_0)}{m_1(s_0)}\right)-\frac{c(\rho_+)\sqrt{q_+^2-c^2(\rho_+)}}{u_{3+}^2-c^2(\rho_+)}\frac{m_2(s_0)}{m_1(s_0)}\right\},$$

$$G_2(s_0)=\left(u_{3+}+\frac{U_+c(\rho_+)}{\sqrt{q_+^2-c^2(\rho_+)}}\right)\left\{\left(1-\frac{u_{3+}U_+}{u_{3+}^2-c^2(\rho_+)}\frac{m_2(s_0)}{m_1(s_0)}\right)+\frac{c(\rho_+)\sqrt{q_+^2-c^2(\rho_+)}}{u_{3+}^2-c^2(\rho_+)}\frac{m_2(s_0)}{m_1(s_0)}\right\}.$$

By a direct simplification, one has

$$G_1(s_0)=u_{3+}-U_+\frac{m_2(s_0)}{m_1(s_0)}-\left(\frac{U_+c(\rho_+)}{\sqrt{q_+^2-c^2(\rho_+)}}+\frac{u_{3+}c(\rho_+)}{\sqrt{q_+^2-c^2(\rho_+)}}\frac{m_2(s_0)}{m_1(s_0)}\right),$$
$$G_2(s_0)=u_{3+}-U_+\frac{m_2(s_0)}{m_1(s_0)}+\left(\frac{U_+c(\rho_+)}{\sqrt{q_+^2-c^2(\rho_+)}}+\frac{u_{3+}c(\rho_+)}{\sqrt{q_+^2-c^2(\rho_+)}}\frac{m_2(s_0)}{m_1(s_0)}\right).$$

Therefore, by use of Lemma 2.4 and Lemma 3.2, we have

$$G_1(s_0)=\frac{q_0}{2}\left\{1-\frac{2-\gamma}{\gamma-1}y_0-\left(1+\frac{1}{2}\left(\frac{3-\gamma}{\gamma-1}-3b_0^2\right)y_0\right)\frac{c_+}{b_0\sqrt{q_+^2-c_+^2}}\right\}+O\left(q_0^{-\frac{2\gamma}{\gamma-1}}\right)+O\left(q_0^{-\frac{4}{\gamma-1}}\right),$$

$$G_2(s_0)=\frac{q_0}{2}\left\{1-\frac{2-\gamma}{\gamma-1}y_0+\left(1+\frac{1}{2}\left(\frac{3-\gamma}{\gamma-1}-3b_0^2\right)y_0\right)\frac{c_+}{b_0\sqrt{q_+^2-c_+^2}}\right\}+O\left(q_0^{-\frac{2\gamma}{\gamma-1}}\right)+O\left(q_0^{-\frac{4}{\gamma-1}}\right).$$

This derives the precise form of $A(s_0)$ in (3.13). By Lemma 2.2 and Lemma 2.4, a direct computation yields the rough form (3.14). □

**Proof of Lemma 3.5.** For the notational convenience, we denote by $\theta=\frac{1}{\Lambda}$, where $\Lambda$ is given in (2.8). By using (2.10), (3.1) and the expressions of $l_1(s_0)$, $l_2(s_0)$, we have

$$\begin{cases} l_1(s_0)=\dfrac{1}{2(1-\theta)^2}\left(\left(1+s_0^2\right)\left(\theta^2-\theta\right)-\dfrac{(1+s_0^2)^2}{q_0s_0^2}\dfrac{U_+c(\rho_+)}{\sqrt{q_+^2-c^2(\rho_+)}}\right), \\ l_2(s_0)=\dfrac{1}{2(1-\theta)^2}\left(\left(1+s_0^2\right)\left(\theta^2-\theta\right)+\dfrac{(1+s_0^2)^2}{q_0s_0^2}\dfrac{U_+c(\rho_+)}{\sqrt{q_+^2-c^2(\rho_+)}}\right). \end{cases} \tag{A.1}$$

We rewrite $A_1(s_0)$ as

$$A_1(s_0)=-\frac{A_{11}(s_0)}{A_{12}(s_0)}$$

with

$$A_{11}(s_0)=\frac{\rho_+'(s_0)U_+^2}{\rho_+}+2U_+U_+'(s_0)+\frac{U_+u_{3+}'(s_0)}{s_0}+(u_{3+}-q_0)\left(\frac{\rho_+'(s_0)u_{3+}}{\rho_+}+u_{3+}'(s_0)\right),$$
$$A_{12}(s_0)=\frac{m_1(s_0)}{\rho_+}\partial_{\omega_1}U_+(s_0)+\frac{m_2(s_0)}{\rho_+}\partial_{\omega_1}u_{3+}(s_0).$$

Next we compute $A_{11}(s_0)$ and $A_{12}(s_0)$ respectively.

It follows from (2.1), (3.1), (2.10) and a direct computation that

$$\begin{aligned} A_{12}(s_0) &= \frac{1}{2}U_+\left(u_{3+}+q_0-\frac{U_+}{s_0}\right)+\frac{q_+^2(q_0u_{3+}-q_+^2)}{2c(\rho_+)\sqrt{q_+^2-c^2(\rho_+)}}+\frac{c(\rho_+)(2U_+^2+(u_{3+}-q_0+\frac{U_+}{s_0})u_{3+})}{2\sqrt{q_+^2-c^2(\rho_+)}} \\ &= \frac{s_0q_0^2}{2(1+s_0^2)}\left(1-\theta^2\right)+\frac{\theta(1-\theta)(1+s_0^2\theta^2)}{2c(\rho_+)\sqrt{q_+^2-c^2(\rho_+)}}\frac{s_0^2q_0^4}{(1+s_0^2)^2} \\ &\quad+\frac{c(\rho_+)(1-\theta)(1-s_0^2\theta)}{2\sqrt{q_+^2-c^2(\rho_+)}}\frac{q_0^2}{1+s_0^2}. \end{aligned} \tag{A.2}$$

Additionally, by the system (2.1) and (2.10), one can get

$$\begin{aligned} U_+'(s_0) &= -\frac{q_0(1-\theta)}{(1+s_0^2)^2}\left(1+\frac{s_0^2\theta^2}{(\gamma-1)\left(\frac{s_0^2}{2}(1-\theta^2)+h(\rho_0)\frac{1+s_0^2}{q_0^2}\right)-s_0^2\theta^2}\right), \\ u_{3+}'(s_0) &= -s_0U_+'(s_0), \\ \frac{\rho_+'(s_0)}{\rho_+} &= -\frac{s_0\theta(1-\theta)}{(1+s_0^2)\left((\gamma-1)\left(\frac{s_0^2}{2}(1-\theta^2)+h(\rho_0)\frac{1+s_0^2}{q_0^2}\right)-s_0^2\theta^2\right)}. \end{aligned}$$

Thus, we have

$$\begin{aligned} A_{11}(s_0) &= \frac{\rho_+'(s_0)}{\rho_+}\left(U_+^2+u_{3+}^2-q_0u_{3+}\right)+U_+'(s_0)\left(U_+-s_0(u_{3+}-q_0)\right) \\ &= \frac{\rho_+'(s_0)}{\rho_+}\frac{s_0^2q_0^2}{1+s_0^2}\left(\theta^2-\theta\right)+U_+'(s_0)s_0q_0(1-\theta) \\ &= -\frac{s_0q_0^2}{(1+s_0^2)^2}\left(1-\theta^2\right). \end{aligned} \tag{A.3}$$

Combining (A.1), (A.2) and (A.3) with Lemma 2.4 yields the precise form (3.15) and the rough form (3.16). Therefore, Lemma 3.5 is proved. □

**Proof of Lemma 3.6.** In view of the fact that $U(b_0)=b_0u_3(b_0)$ and the expression of $B(b_0)$, a direct computation yields

$$B(b_0) = -\frac{B_{11}(b_0)}{B_{12}(b_0)}$$

with

$$\begin{aligned} B_{11}(b_0) &= \left\{\left(1+\frac{b_0^2u_3^2(b_0)}{u_3^2(b_0)-c^2(b_0)}\right)-\frac{b_0c(b_0)\sqrt{q^2(b_0)-c^2(b_0)}}{u_3^2(b_0)-c^2(b_0)}\right\}\left(1+\frac{b_0c(\rho(b_0))}{\sqrt{q^2(b_0)-c^2(b_0)}}\right), \\ B_{12}(b_0) &= \left\{\left(1+\frac{b_0^2u_3^2(b_0)}{u_3^2(b_0)-c^2(b_0)}\right)+\frac{b_0c(b_0)\sqrt{q^2(b_0)-c^2(b_0)}}{u_3^2(b_0)-c^2(b_0)}\right\}\left(1-\frac{b_0c(\rho(b_0))}{\sqrt{q^2(b_0)-c^2(b_0)}}\right); \end{aligned}$$

here $c(b_0)=c(\rho(b_0))$.

Moreover, it follows from $U(b_0) = b_0 u_3(b_0)$ and a direct computation that

$$B_{11}(b_0) = B_{12}(b_0).$$

This yields

$$B(b_0) = -1. \qquad \square$$

## Appendix B

In order to give the estimates on $g_i$ $(i = 1, 2)$ and their first order derivatives in (3.6), we require the precise computations on $f_i(\hat{\omega}(\frac{r}{x_3}))$ and $\lambda_i(\hat{\omega}(\frac{r}{x_3}))$ (that is, we replace $\omega(x_3, r)$ by $\hat{\omega}(\frac{r}{x_3})$ in the expressions of $f_i(\omega(x_3, r))$ and $\lambda_i(\omega(x_3, r))$).

**Lemma B.1.** *Denoted by $f_i(s) = f_i(\hat{\omega}(s))$ and $\lambda_i(s) = \lambda_i(\hat{\omega}(s))$ with $s = \frac{r}{x_3}$, $\tau_0 > 0$ is given in Remark 2.3. Then for $1 < \gamma < 3$, $b_0 \leqslant s \leqslant s_0 + \tau_0$ and large $q_0$, we have*

$$\hat{\omega}_i'(s) = -\frac{1}{1 + b_0^2} + O\big(q_0^{-2}\big) + O\big(q_0^{-\frac{2}{\gamma-1}}\big),$$

$$\hat{\omega}_1''(s) = \frac{2}{b_0(1 + b_0^2)} + \frac{2}{b_0(1 + b_0^2)^2}\sqrt{\frac{2 - (\gamma - 1)b_0^2}{\gamma - 1}} + O\big(q_0^{-2}\big) + O\big(q_0^{-\frac{2}{\gamma-1}}\big),$$

$$\hat{\omega}_2''(s) = \frac{2}{b_0(1 + b_0^2)} - \frac{2}{b_0(1 + b_0^2)^2}\sqrt{\frac{2 - (\gamma - 1)b_0^2}{\gamma - 1}} + O\big(q_0^{-2}\big) + O\big(q_0^{-\frac{2}{\gamma-1}}\big),$$

$$\partial_{\omega_i} f_j(s) = C(b_0, \gamma) + O\big({q_0}^{-2}\big) + O\big(q_0^{-\frac{2}{\gamma-1}}\big), \quad i, j = 1, 2,$$

$$\partial_{\omega_i} \lambda_j(s) = C(b_0, \gamma) + O\big({q_0}^{-2}\big) + O\big(q_0^{-\frac{2}{\gamma-1}}\big), \quad i, j = 1, 2;$$

*here the generic constant $C(b_0, \gamma)$ depends only on $\gamma$ and $b_0$, but is independent of $q_0$.*

**Remark B.1.** By Lemma B.1, we can derive that for large $q_0$ and sufficiently small $\varepsilon$, the source terms in the system (3.5) can be estimated as follows

$$\big|g_i\big(W(x_3, r)\big)\big| \leqslant \big(C + O\big(q_0^{-\frac{2}{\gamma-1}}\big) + O(\varepsilon)\big)\frac{1}{x_3}\sum_{j=1}^{2}\big|W_j(x_3, r)\big|, \quad i = 1, 2. \tag{B.1}$$

**Proof.** Since $\hat{\omega}(s)$ is the extension of $\omega(s)$ in $[b_0, s_0 + \tau_0]$ and $\tau_0 < q_0^{-\frac{4}{\gamma-1}}$ holds, then it is enough in our computations to use $\omega(s)$ instead of $\hat{\omega}(s)$.

By the expression of $\omega_1(s)$, we have

$$\omega_1'(s) = \frac{U'(s)u_3(s) - U(s)u_3'(s)}{q^2(s)} + \frac{\sqrt{q^2(s) - c^2(\rho(s))}}{q^2(s)c(\rho(s))}\big(U(s)U'(s) + u_3(s)u_3'(s)\big).$$

Since by use of Lemma 2.2 one can get

$$\begin{cases} U'(s)u_3(s) - U(s)u_3'(s) = -\dfrac{q_0^2}{(1+b_0^2)^2}\big(1 + O\big(q_0^{-2}\big) + O\big(q_0^{-\frac{2}{\gamma-1}}\big)\big), \\ \sqrt{q^2(s) - c^2\big(\rho(s)\big)} = q_0\sqrt{\dfrac{2-(\gamma-1)b_0^2}{2(1+b_0^2)}}\big(1 + O\big(q_0^{-2}\big) + O\big(q_0^{-\frac{2}{\gamma-1}}\big)\big), \\ U(s)U'(s) + u_3(s)u_3'(s) = q_0^2\big(O\big(q_0^{-2}\big) + O\big(q_0^{-\frac{2}{\gamma-1}}\big)\big), \end{cases} \tag{B.2}$$

then by (B.2), a direct computation yields

$$\omega_1'(s) = -\frac{1}{1+b_0^2} + O\big(q_0^{-2}\big) + O\big(q_0^{-\frac{2}{\gamma-1}}\big).$$

Similarly, one has

$$\omega_2'(s) = -\frac{1}{1+b_0^2} + O\big(q_0^{-2}\big) + O\big(q_0^{-\frac{2}{\gamma-1}}\big).$$

In addition, by Lemma 2.2, Lemma 2.3 and (B.1), a direct computation yields the estimates of $\hat{\omega}_1''(s)$ and $\hat{\omega}_2''(s)$ in Lemma B.1.

Next, we estimate $\partial_{\omega_i} f_j(\omega(s))$ and $\partial_{\omega_i}\lambda_j(\omega(s))$, $i, j = 1, 2$.

By the Bernoulli's law (1.5) and (3.1), we can arrive at

$$\begin{cases} \partial_{\omega_1} c = -\dfrac{\gamma-1}{4}\dfrac{q^2}{\sqrt{q^2-c^2}}, \\ \partial_{\omega_1}\sqrt{q^2-c^2} = \dfrac{\gamma+1}{4}\dfrac{q^2 c}{q^2-c^2}. \end{cases} \tag{B.3}$$

Since it follows from the expressions of $f_1(\omega(s))$ and $\lambda_2(\omega(s))$ that

$$\partial_{\omega_1} f_1\big(\omega(s)\big) = \frac{cU(\sqrt{q^2-c^2}\,\partial_{\omega_1}u_3 + u_3\partial_{\omega_1}\sqrt{q^2-c^2}) - \partial_{\omega_1}(cU)u_3\sqrt{q^2-c^2}}{(cU - u_3\sqrt{q^2-c^2})^2},$$

$$\partial_{\omega_1}\lambda_2 = \frac{\partial_{\omega_1}(u_3U + c\sqrt{q^2-c^2})}{{u_3}^2 - c^2} - \frac{(u_3U + c\sqrt{q^2-c^2})(2u_3\partial_{\omega_1}u_3 - 2c\partial_{\omega_1}c)}{({u_3}^2-c^2)^2}$$

with $c = c(\rho(s))$ and $q^2 = u_3^2 + U^2$.

Then, from Lemma 2.2, (B.3), (3.1) and a direct computation, one has

$$\partial_{\omega_1} f_1(s) = C(b_0, \gamma) + O\big({q_0}^{-2}\big) + O\big(q_0^{-\frac{2}{\gamma-1}}\big),$$

$$\partial_{\omega_1}\lambda_2(s) = C(b_0, \gamma) + O\big({q_0}^{-2}\big) + O\big(q_0^{-\frac{2}{\gamma-1}}\big).$$

By the analogous method, we can compute the other terms in Lemma B.1. $\square$

**Lemma B.2.** *Under the assumptions in Lemma* B.1, *we have*

$$\partial^2_{\omega_i\omega_j} f_k(s) = C(b_0,\gamma) + O\big({q_0}^{-2}\big) + O\big(q_0^{-\frac{2}{\gamma-1}}\big),$$

$$\partial^2_{\omega_i\omega_j} \lambda_k(s) = C(b_0,\gamma) + O\big({q_0}^{-2}\big) + O\big(q_0^{-\frac{2}{\gamma-1}}\big)$$

*for* $i, j, k = 1, 2$.

The proof comes from a direct computation, here we omit it.

**Lemma B.3.** *Under the assumptions in* (4.1), *for small* $\varepsilon$ *and large* $q_0$, *we have*

$$\begin{cases} d_i - C(b_0,\gamma)\varepsilon \leqslant \dfrac{\Gamma_i(x_3,r)}{x_3} \leqslant 1 + C(b_0,\gamma)\varepsilon, & i = 1, 2, \\ d_1 d_2 - C(b_0,\gamma)\varepsilon \leqslant \dfrac{\pi_i(x_3,r)}{x_3} \leqslant d_i + C(b_0,\gamma)\varepsilon, & i = 1, 2, \end{cases} \tag{B.4}$$

*where* $d_1 = \frac{b_0-\lambda_1(s)}{s_0-\lambda_1(s)}$, $d_2 = \frac{\lambda_2(s)-s_0}{\lambda_2(s)-b_0}$, *and* $1 < \frac{1}{d_1} < 1 + C(b_0,\gamma)q_0^{-\frac{2}{\gamma-1}}$, $1 < \frac{1}{d_1 d_2} < 1 + C(b_0,\gamma)q_0^{-\frac{2}{\gamma-1}}$.

**Proof.** First, it follows from (4.2) that

$$\xi_1(x_3,r) - r = \int_{x_3}^{\Gamma_1} \lambda_1\big(\omega\big(t, \gamma_1(t,x_3,r)\big)\big)\,dt. \tag{B.5}$$

In addition, by the assumptions in (4.1) we have

$$\left|\frac{\chi(t) - s_0 t}{t}\right| \leqslant C\varepsilon. \tag{B.6}$$

Since

$$\xi_1(x_3,r) - s_0\Gamma_1(x_3,r) + s_0\Gamma_1(x_3,r) - r = \int_{x_3}^{\Gamma_1} \big(\lambda_1\big(\omega\big(t,\gamma_1(t,x_3,r)\big)\big) - \lambda_1(s)\big)\,dt + \lambda_1(s)(\Gamma_1 - x_3)$$

for $(x_3,r) \in D_T$, then by use of (B.6) and Lemma B.1 we have

$$\big(s_0 - \lambda_1(s)\big)\frac{\Gamma_1(x_3,r)}{x_3} = \frac{r}{x_3} - \lambda_1(s) + O(\varepsilon);$$

this, together with $b_0 < \frac{r}{x_3} < s_0 + C\varepsilon$, yields the estimate of $\frac{\Gamma_1(x_3,r)}{x_3}$ in Lemma B.3.

Similarly, noting that $\eta_2(x_3,r) - \xi_1(x_3,r) = \int_{\Gamma_1(x_3,r)}^{\pi_2(x_3,r)} \lambda_2(\omega(t,\gamma_2(t,\Gamma_1,\xi_1)))\,dt$ and $\eta_2(x_3,r) = b_0\pi_2(x_3,r)$ hold, we have

$$\big(\lambda_2(s) - b_0\big)\frac{\pi_2(x_3,r)}{\Gamma_1(x_3,r)} = \lambda_2(s) - \frac{\xi_1(x_3,r)}{\Gamma_1(x_3,r)} + O(\varepsilon).$$

This implies

$$\frac{\pi_2(x_3,r)}{\Gamma_1(x_3,r)}=\frac{\lambda_2(s)-s_0}{\lambda_2(s)-b_0}+O(\varepsilon).$$

Together with the estimate of $\frac{\Gamma_1(x_3,r)}{x_3}$ above, we can obtain the estimate of $\frac{\pi_2(x_3,r)}{x_3}$ as asserted in Lemma B.3.

In analogous way, one can estimate $\frac{\Gamma_2(x_3,r)}{x_3}$ and $\frac{\pi_1(x_3,r)}{x_3}$.

Finally, we treat the terms $\frac{1}{d_1}$ and $\frac{1}{d_1 d_2}$.

For large $q_0$ and $0<b_0<b_*$, a direct computation yields

$$\begin{aligned}\lambda_1(s)-b_0&=\frac{\sqrt{\frac{\gamma-1}{2}}b_0(\frac{\gamma-1}{2}b_0^4+\frac{\gamma-1}{2}b_0^2-1)}{(\frac{1}{1+b_0^2}-\frac{\gamma-1}{2}b_0^2)(\sqrt{\frac{\gamma-1}{2}}b_0^2+\sqrt{1-\frac{\gamma-1}{2}b_0^2})}\bigl(1+O\bigl(q_0^{-2}\bigr)+O\bigl(q_0^{-\frac{2}{\gamma-1}}\bigr)\bigr)\\&=-\frac{b_0(1+b_0^2)\sqrt{\frac{\gamma-1}{2}}}{\sqrt{\frac{\gamma-1}{2}}b_0^2+\sqrt{1-\frac{\gamma-1}{2}b_0^2}}\bigl(1+O\bigl(q_0^{-2}\bigr)+O\bigl(q_0^{-\frac{2}{\gamma-1}}\bigr)\bigr)<0.\end{aligned}\tag{B.7}$$

Thus, it follows from the expression of $d_1$ that $\frac{1}{d_1}>1$ holds.

Similarly, one has $\frac{1}{d_1 d_2}>1$.

In addition, by Lemma 2.2, a direct computation yields

$$\begin{aligned}\frac{1}{d_1 d_2}-1&=\frac{(s_0-b_0)(\lambda_2(s)-\lambda_1(s))}{(b_0-\lambda_1(s))(\lambda_2(s)-s_0)}\leqslant C(b_0,\gamma)q_0^{-\frac{2}{\gamma-1}},\\ \frac{1}{d_1}-1&=\frac{s_0-b_0}{b_0-\lambda_1(s)}\leqslant C(b_0,\gamma)q_0^{-\frac{2}{\gamma-1}}.\end{aligned}$$

Therefore, we complete the proof of Lemma B.3. □

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